

Aries: A Three-Segment Path for Solving Trust-Region Subproblems

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Abstract

Trust-region methods are fundamental to nonlinear optimization, but solving the trust-region subproblem exactly can be computationally expensive. This paper proposes Aries, a three-segment path approximation that extends the classical dogleg path while retaining comparable per-iteration work. We state the assumptions under which the usual trust-region sufficient-decrease argument applies and describe an adaptive rule for selecting the path parameters. Numerical experiments on the S2MPJ test set using Dolan–Moré performance profiles indicate that adaptive Aries is competitive with both fixed-parameter Aries and dogleg, with the precise ranking depending on whether CPU time or iteration count is measured.

1 Introduction

Trust-region methods are widely used in nonlinear optimization because of their robustness and reliable global behavior [11, 1, 14]. At iteration k , the objective is approximated by a quadratic model in a neighborhood of the current iterate, and the model is minimized subject to a trust-region constraint to obtain a step d_k . The matrix B_k is typically updated via a quasi-Newton method such as BFGS [4, 7, 8, 13]. The trust-region subproblem is given by:

$$\begin{aligned} d_k = \arg \min_{p \in \mathbb{R}^n} \quad & m_k(p) := \frac{1}{2}p^\top B_k p + g_k^\top p + f_k \\ \text{s.t.} \quad & \|p\| \leq \Delta_k \end{aligned} \tag{1}$$

Here, g_k is the gradient, B_k is a symmetric positive definite matrix, f_k is the current objective value, and $\Delta_k > 0$ is the trust-region radius. In implementations that approximate the Hessian using gradient information, BFGS is a standard quasi-Newton choice; its independent 1970 contributions are due to Broyden, Fletcher, Goldfarb, and Shanno [4, 7, 8, 13]. Exact solution of this problem can be costly, so practical methods use inexpensive approximations. The dogleg path, classically

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associated with Powell's hybrid method [12], joins the Cauchy point and the Newton step by two line segments; its use for approximately solving trust-region subproblems is described by Nocedal and Wright [11]. Other inexpensive subproblem solvers have also been considered: Kouri [10] proposed a spectral projected gradient (SPG) solver for the trust-region subproblem, and Baraldi and Kouri [2] compared the performance of several subproblem solvers, including SPG and dogleg. We propose Aries, a three-segment path that inserts an interpolating segment between truncated portions of the dogleg path.

The structure of this work is as follows. In Section 2, we describe the algorithmic implementation of Aries. In Section 3, we establish properties of the path that guarantee sufficient decrease in the model function. In Section 4, we describe an adaptive rule for selecting the parameters α and β . Finally, in Section 5, we present numerical results on the S2MPJ test set comparing the dogleg method with Aries.

2 The Aries algorithm

Let

$$p_k^U = -\frac{g_k^\top g_k}{g_k^\top B_k g_k} g_k, \quad p_k^B = -B_k^{-1} g_k. \quad (2)$$

For $g_k \neq 0$ and parameters $\alpha, \beta \in (0, 1)$, the Aries path is parametrized by

$$d_k(\tau) = \begin{cases} \tau p_k^U, & \text{if } 0 \leq \tau \leq \alpha, \\ \left(\frac{\tau - \alpha}{1 + \beta - \alpha}\right) (\beta p_k^B + (1 - \beta - \alpha) p_k^U) + \alpha p_k^U, & \text{if } \alpha < \tau \leq 1 + \beta, \\ p_k^U + (\tau - 1)(p_k^B - p_k^U), & \text{if } 1 + \beta < \tau \leq 2, \end{cases} \quad (3)$$

where the first and third segments agree at their shared endpoints with the corresponding truncated dogleg segments. The parameter α controls the length of the initial steepest-descent segment, while β controls the weight assigned to the Newton step in the final part of the path.

The Aries method is very similar in structure to the dogleg method, with the addition of an interpolating segment between the Cauchy point and the Newton step. Therefore, the algorithm requires the same Cauchy-point and Newton-step computations as dogleg. As shown in Section 3, when the Newton step is infeasible, the approximate solution of the subproblem can be found by intersecting the Aries path (3) with the trust-region boundary. The full algorithm is described in Algorithm 1.

3 Model-decrease properties

For a trust-region method to converge, the approximate solution of the subproblem must yield sufficient decrease in the model. The dogleg method satisfies this property, and we show next that Aries does as well. First, we show that the model function is strictly decreasing along the Aries path.

Theorem 1. *Suppose that B_k is symmetric positive definite, $g_k \neq 0$, and $\alpha, \beta \in (0, 1)$. Then the model function m_k is strictly decreasing along the Aries path (3).*

Algorithm 1: Aries step computation.

```

Input:  $\Delta_k, g_k, B_k, \alpha_k, \beta_k$ 
Output:  $d_k$ 
1  $p_k^U = -\frac{g_k^\top g_k}{g_k^\top B_k g_k} g_k$ ;
2  $v = \alpha_k p_k^U$ ;
3 if  $\|v\|_2 > \Delta_k$  then
    /* First segment */
4      $d_k \leftarrow \Delta_k \frac{p_k^U}{\|p_k^U\|}$ ;
5 else
6     Solve  $B_k p_k^B = -g_k$  for  $p_k^B$ ;
7     if  $\|p_k^B\|_2 \leq \Delta_k$  then
        /* Newton's step is feasible */
8          $d_k \leftarrow p_k^B$ ;
9     else if  $\|\beta_k(p_k^B - p_k^U) + p_k^U\|_2 > \Delta_k$  then
        /* Second segment */
10         $w = \beta_k p_k^B + (1 - \alpha_k - \beta_k)p_k^U$ ;
11         $p_1 = w^\top w$ ;
12         $p_2 = w^\top v$ ;
13         $p_3 = (\Delta_k^2 - v^\top v)$ ;
14         $\tau = \frac{-p_2 + \sqrt{p_2^2 + p_1 p_3}}{p_1}$ ;
15         $d_k \leftarrow v + \tau w$ ;
16    else
        /* Third segment */
17         $w = p_k^B - p_k^U$ ;
18         $p_1 = w^\top w$ ;
19         $p_2 = w^\top p_k^U$ ;
20         $p_3 = (\Delta_k^2 - p_k^{U\top} p_k^U)$ ;
21         $\tau = \frac{-p_2 + \sqrt{p_2^2 + p_1 p_3}}{p_1}$ ;
22         $d_k \leftarrow p_k^U + \tau w$ ;
23    end
24 end

```

Proof. The first and third segments of Aries' path coincide with the dogleg path, which has been shown to strictly decrease the model function in [11].

For the second segment, we can rewrite d_k as:

$$\begin{aligned} d_k(t) &= tw + c, & w &= -(\beta B_k^{-1} + (1 - \beta - \alpha)\gamma I)g_k, \\ c &= -\alpha\gamma g_k, & \gamma &= \frac{g_k^\top g_k}{g_k^\top B_k g_k}. \end{aligned} \tag{4}$$

Define $u(t) := m_k(0) - m_k(d_k(t))$ as the decrease in the model function by the second segment, then:

$$u(t) = -\frac{1}{2}t^2 g_k^\top (\beta B_k^{-1} + (1 - \beta - \alpha)\gamma I) B_k (\beta B_k^{-1} + (1 - \beta - \alpha)\gamma I) g_k \tag{5}$$

$$\begin{aligned} &- t g_k^\top (\beta B_k^{-1} + (1 - \beta - \alpha)\gamma I) B_k (\alpha\gamma I - B_k^{-1}) g_k \\ &- \frac{1}{2}\alpha^2\gamma^2 g_k^\top B_k g_k + \alpha\gamma g_k^\top g_k, \\ &= -\frac{1}{2}t^2 g_k^\top (\beta^2 B_k^{-1} + 2\beta(1 - \beta - \alpha)\gamma I + (1 - \beta - \alpha)^2\gamma^2 B_k) g_k \\ &+ t g_k^\top (-\alpha\beta\gamma I + \beta B_k^{-1} - \alpha\gamma^2(1 - \beta - \alpha)B_k + \gamma(1 - \beta - \alpha)I) g_k \\ &- g_k^\top \left(\frac{1}{2}\alpha^2\gamma^2 B_k - \alpha\gamma I \right) g_k, \end{aligned} \tag{6}$$

$$\begin{aligned} &= -\frac{1}{2}t^2 g_k^\top (\beta^2 B_k^{-1} + 2\beta\gamma(1 - \beta - \alpha)I + (1 - \beta - \alpha)^2\gamma^2 B_k) g_k \\ &+ t g_k^\top (\beta B_k^{-1} + \gamma(1 - \beta - \alpha - \alpha\beta)I - \alpha\gamma^2(1 - \beta - \alpha)B_k) g_k \\ &+ \alpha\gamma g_k^\top \left(I - \frac{1}{2}\alpha\gamma B_k \right) g_k. \end{aligned} \tag{7}$$

Substituting $\gamma = \frac{g_k^\top g_k}{g_k^\top B_k g_k}$, we have:

$$u(t) = -\frac{1}{2}t^2 \left(\beta^2 g_k^\top B_k^{-1} g_k + 2\beta(1-\beta-\alpha) \frac{(g_k^\top g_k)^2}{g_k^\top B_k g_k} + (1-\beta-\alpha)^2 \frac{(g_k^\top g_k)^2}{g_k^\top B_k g_k} \right) \quad (8)$$

$$+ t \left(\beta g_k^\top B_k^{-1} g_k + (1-\beta-\alpha-\alpha\beta) \frac{(g_k^\top g_k)^2}{g_k^\top B_k g_k} - \alpha(1-\beta-\alpha) \frac{(g_k^\top g_k)^2}{g_k^\top B_k g_k} \right) \\ + \alpha \frac{(g_k^\top g_k)^2}{g_k^\top B_k g_k} \left(1 - \frac{1}{2}\alpha \right), \quad (9)$$

$$= -\frac{1}{2}t^2 \left(\beta^2 g_k^\top B_k^{-1} g_k + ((1-\alpha)^2 - \beta^2) \frac{(g_k^\top g_k)^2}{g_k^\top B_k g_k} \right) \quad (9)$$

$$+ t \left(\beta g_k^\top B_k^{-1} g_k + (1-2\alpha-\beta+\alpha^2) \frac{(g_k^\top g_k)^2}{g_k^\top B_k g_k} \right) \\ + \alpha \frac{(g_k^\top g_k)^2}{g_k^\top B_k g_k} \left(1 - \frac{1}{2}\alpha \right), \quad (10)$$

$$= -\frac{1}{2}t^2 \frac{(g_k^\top g_k)^2}{g_k^\top B_k g_k} \left(\left(\frac{g_k^\top B_k g_k g_k^\top B_k^{-1} g_k}{(g_k^\top g_k)^2} - 1 \right) \beta^2 + (1-\alpha)^2 \right) \quad (10)$$

$$+ t \frac{(g_k^\top g_k)^2}{g_k^\top B_k g_k} \left(\left(\frac{g_k^\top B_k g_k g_k^\top B_k^{-1} g_k}{(g_k^\top g_k)^2} - 1 \right) \beta + (1-\alpha)^2 \right) \\ + \alpha \frac{(g_k^\top g_k)^2}{g_k^\top B_k g_k} \left(1 - \frac{1}{2}\alpha \right),$$

Thus, $u(t)$ is a parabola:

$$u(t) = \frac{1}{2}Qt^2 + Lt + R, \quad (11)$$

$$Q = -\frac{(g_k^\top g_k)^2}{g_k^\top B_k g_k} \left(\left(\frac{g_k^\top B_k g_k g_k^\top B_k^{-1} g_k}{(g_k^\top g_k)^2} - 1 \right) \beta^2 + (1-\alpha)^2 \right), \quad (12)$$

$$L = \frac{(g_k^\top g_k)^2}{g_k^\top B_k g_k} \left(\left(\frac{g_k^\top B_k g_k g_k^\top B_k^{-1} g_k}{(g_k^\top g_k)^2} - 1 \right) \beta + (1-\alpha)^2 \right), \quad (13)$$

$$R = \alpha \frac{(g_k^\top g_k)^2}{g_k^\top B_k g_k} \left(1 - \frac{1}{2}\alpha \right). \quad (14)$$

Note that $u(t)$ is a concave parabola in t since $Q < 0$, with $u(0) = R > 0$. Thus, it suffices to show that this parabola reaches its maximum at a point greater than or equal to 1. The vertex t_v of the parabola $\frac{1}{2}Qt^2 + Lt + R$ is given by $\frac{L}{-Q} > 0$. Since $\beta \in (0, 1)$,

$$\beta - \beta^2 > 0, \quad (15)$$

and by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} g_k^\top B_k g_k g_k^\top B_k^{-1} g_k &\geq (g_k^\top g_k)^2 \\ \frac{g_k^\top B_k g_k g_k^\top B_k^{-1} g_k}{(g_k^\top g_k)^2} - 1 &\geq 0. \end{aligned} \quad (16)$$

Multiplying (16) and (15)

$$\begin{aligned} \left(\frac{g_k^\top B_k g_k g_k^\top B_k^{-1} g_k}{(g_k^\top g_k)^2} - 1 \right) (\beta - \beta^2) &\geq 0, \\ \left(\frac{g_k^\top B_k g_k g_k^\top B_k^{-1} g_k}{(g_k^\top g_k)^2} - 1 \right) \beta &\geq \left(\frac{g_k^\top B_k g_k g_k^\top B_k^{-1} g_k}{(g_k^\top g_k)^2} - 1 \right) \beta^2, \\ \left(\frac{g_k^\top B_k g_k g_k^\top B_k^{-1} g_k}{(g_k^\top g_k)^2} - 1 \right) \beta + (1 - \alpha)^2 &\geq \left(\frac{g_k^\top B_k g_k g_k^\top B_k^{-1} g_k}{(g_k^\top g_k)^2} - 1 \right) \beta^2 + (1 - \alpha)^2 > 0 \\ L &\geq -Q > 0, \\ \frac{L}{-Q} &\geq 1 \end{aligned} \quad (17)$$

Therefore, $u(t)$ is strictly increasing in the interval $[0, 1]$, and is positive in this interval, which shows that the model function m_k is strictly decreasing along Aries' path. $\square$

Next, we show that the Euclidean norm of d_k is strictly increasing in τ . This property is important for the implementation of Aries because it allows the intersection of the path with the trust-region boundary to be found by solving a scalar quadratic equation, as in the dogleg method.

Theorem 2. *Suppose that B_k is symmetric positive definite, $g_k \neq 0$, and $\alpha, \beta \in (0, 1)$. Then $\|d_k(\tau)\|$ is strictly increasing in τ along the Aries path (3).*

Proof. The first and third segments of Aries' path coincide with the dogleg path, which has been shown to be strictly increasing in norm, as shown in [11].

For the second segment, we have:

$$\|d_k\|^2 = \|tw + c\|^2, \quad (18)$$

$$\|d_k\|^2 = t^2 \|w\|^2 + 2tw^\top c + \|c\|^2. \quad (19)$$

Taking the derivative with respect to t , we have:

$$\frac{d}{dt} \|d_k\|^2 = 2t \|w\|^2 + 2w^\top c, \quad (20)$$

$$\frac{d}{dt} \|d_k\|^2 \geq 2w^\top c, \quad (21)$$

since $t \geq 0$. Now, we shall show that $w^\top c > 0$:

$$w^\top c = g_k^\top (\beta B_k^{-1} + (1 - \beta - \alpha)\gamma I) \alpha \gamma g_k, \quad (22)$$

$$w^\top c = \beta g_k^\top B_k^{-1} g_k \alpha \frac{g_k^\top g_k}{g_k^\top B_k g_k} + \alpha(1 - \beta - \alpha) \frac{(g_k^\top g_k)^3}{(g_k^\top B_k g_k)^2}, \quad (23)$$

$$w^\top c = \alpha \frac{(g_k^\top g_k)^3}{(g_k^\top B_k g_k)^2} \left(\beta \frac{g_k^\top B_k g_k g_k^\top B_k^{-1} g_k}{(g_k^\top g_k)^2} + 1 - \alpha - \beta \right), \quad (24)$$

$$w^\top c > \alpha \frac{(g_k^\top g_k)^3}{(g_k^\top B_k g_k)^2} \left(\beta \frac{g_k^\top B_k g_k g_k^\top B_k^{-1} g_k}{(g_k^\top g_k)^2} - \beta \right), \quad (25)$$

$$w^\top c > \alpha \beta \frac{(g_k^\top g_k)^3}{(g_k^\top B_k g_k)^2} \left(\frac{g_k^\top B_k g_k g_k^\top B_k^{-1} g_k}{(g_k^\top g_k)^2} - 1 \right). \quad (26)$$

By the Cauchy-Schwarz inequality, we have:

$$w^\top c > 0. \quad (27)$$

Therefore, $\frac{d}{dt} \|d_k\|^2 > 0$, which shows that the Euclidean norm of d_k is strictly increasing in τ . $\square$

With these properties, it remains to show that the Aries step satisfies equation (4.20) in [11]:

$$m_k(0) - m_k(d_k) \geq c_1 \|g_k\| \min \left\{ \Delta_k, \frac{\|g_k\|}{\|B_k\|} \right\}, \quad (28)$$

where $c_1 \in (0, 1)$. This is the sufficient decrease condition that guarantees convergence of the trust-region method when the subproblem is solved approximately.

Theorem 3. *Under the assumptions of the preceding theorems, the model reduction achieved by the Aries step satisfies (28) with $c_1 = \frac{\alpha(2-\alpha)}{2}$.*

Proof. Recall from (10) that the model function reduction is given by:

$$u(t) = -\frac{1}{2}t^2 \frac{(g_k^\top g_k)^2}{g_k^\top B_k g_k} \left(\left(\frac{g_k^\top B_k g_k g_k^\top B_k^{-1} g_k}{(g_k^\top g_k)^2} - 1 \right) \beta^2 + (1 - \alpha)^2 \right) \quad (29)$$

$$+ t \frac{(g_k^\top g_k)^2}{g_k^\top B_k g_k} \left(\left(\frac{g_k^\top B_k g_k g_k^\top B_k^{-1} g_k}{(g_k^\top g_k)^2} - 1 \right) \beta + (1 - \alpha)^2 \right) \\ + \alpha \frac{(g_k^\top g_k)^2}{g_k^\top B_k g_k} \left(1 - \frac{1}{2}\alpha \right),$$

$$u(t) \geq \frac{\alpha(2 - \alpha)}{2} \frac{(g_k^\top g_k)^2}{g_k^\top B_k g_k}, \quad (30)$$

$$u(t) \geq \frac{\alpha(2 - \alpha)}{2} \frac{g_k^\top g_k}{\|B_k\|}, \quad (31)$$

$$u(t) \geq \frac{\alpha(2 - \alpha)}{2} \|g_k\| \frac{\|g_k\|}{\|B_k\|}, \quad (32)$$

$$u(t) \geq \frac{\alpha(2 - \alpha)}{2} \|g_k\| \min \left\{ \Delta_k, \frac{\|g_k\|}{\|B_k\|} \right\}. \quad (33)$$

Therefore, the model function reduction achieved by d_k satisfies (28) with $c_1 = \frac{\alpha(2-\alpha)}{2}$. $\square$

Because the quality of the interpolating segment depends on the local geometry of the model, Aries can benefit from adaptively selecting α and β instead of using fixed values. In the following section, we describe a simple adaptive rule based only on current curvature information.

4 Adaptive selection of the parameters

The parameters α and β in the Aries path control the shape of the path and, therefore, the quality of the approximation. Although fixed values may work well, they may require careful tuning for different problems, and the best parameters may change from one iteration to the next. For this reason, we propose the following adaptive strategy, based on the curvature of the model:

$$\beta_k = \frac{g_k^\top B_k g_k}{\|B_k g_k\| \|g_k\|}, \quad (34) \\ \alpha_k = 1 - \beta_k.$$

The adaptive rule may attain the boundary case $\alpha_k = 0$ and $\beta_k = 1$. Equality in the Cauchy-Schwarz inequality then implies $B_k g_k = \lambda g_k$ for some $\lambda > 0$, so $p_k^U = p_k^B$ and the Aries path reduces to the dogleg path. This degenerate case is handled appropriately by the algorithm and does not invoke the strict $\alpha > 0$ bound used in the preceding sufficient-decrease result. Although the rule (34) uses only curvature information available at the current iteration, alternative strategies may incorporate additional information. We considered several parameter-selection strategies in our implementation and found that the rule (34) provided the best overall performance among those tested. The implementation and the alternative strategies considered in our experiments are publicly available in the Aries repository (see Section 5).

5 Numerical results

We compare dogleg with Aries using fixed parameters $\alpha = 1/4$ and $\beta = 1/2$, and with the adaptive rule in (34). The test problems are taken from S2MPJ [9], and the experiments are implemented in Julia [3]. The Aries repository, which hosts the implementation and tested alternatives, is available at <https://github.com/JMSaavedraA/ARIES>. Both methods share the same trust-region framework and stopping criteria; only the subproblem approximation differs. Performance profiles for CPU time and iteration count are shown in Figure 1, following Dolan and Moré [6].

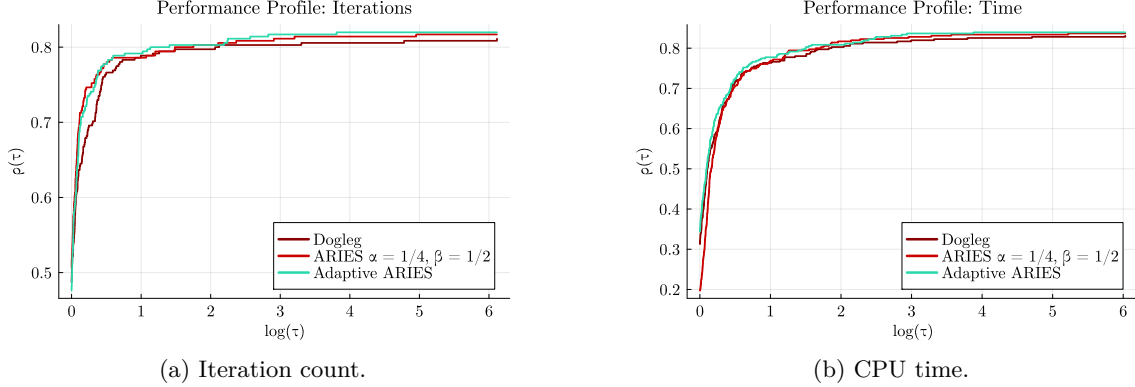


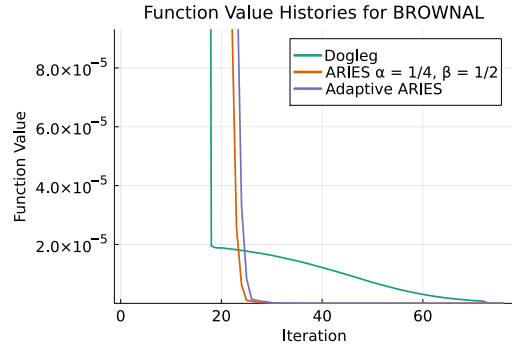
Figure 1: Dolan–Moré performance profiles comparing dogleg with Aries using fixed and adaptive parameters.

The time profile suggests that adaptive Aries is the most frequently fastest method: $\rho(1) \approx 0.35$, compared with approximately 0.31 for dogleg and 0.20 for fixed-parameter Aries. The adaptive curve remains competitive over the range of performance factors displayed. As τ increases, the curves approach solve fractions of approximately 0.845, 0.835, and 0.825 for adaptive Aries, fixed Aries, and dogleg, respectively. These percentages describe this experiment only; they are not universal performance guarantees.

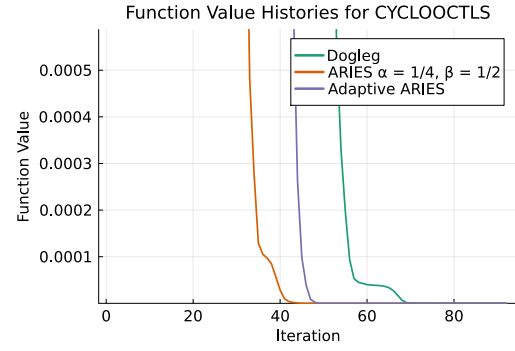
The iteration profile shows dogleg and fixed-parameter Aries tied or nearly tied at the left of the plot, with adaptive Aries initially slightly lower. Adaptive Aries becomes the leading curve for moderate performance factors, while fixed-parameter Aries remains competitive with dogleg. The limiting solve fractions are approximately 0.845, 0.835, and 0.815–0.820 for adaptive Aries, fixed Aries, and dogleg, respectively. Again, these observations are specific to the reported test set and experimental protocol.

To illustrate the behavior on specific problems where Aries outperforms dogleg, Figure 2 shows the evolution of the function values for several examples from the S2MPJ set. The CPU times for these functions are summarized in Table 1, and the iteration counts are provided in Table 2.

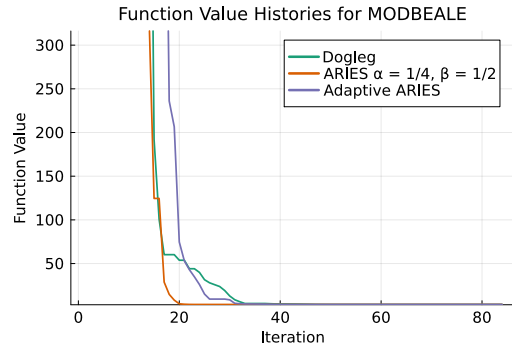
The CPU-time reductions from Dogleg to Aries are substantial for the majority of problems. The median speed-up, computed as $\frac{\text{Dogleg}}{\text{Aries}}$, is **1.35** $\times$ (see the bolded median values in Table 1). Six of the eight benchmark functions achieve a speed-up greater than $1.2\times$, with the most pronounced gain observed for **OSBORNEA** (approximately $1.9\times$ faster). The outlier **PFIT3LS** shows a slight slowdown ($\approx 1.04\times$ slower), which can be attributed to the larger problem size and the overhead of the additional segment in Aries.



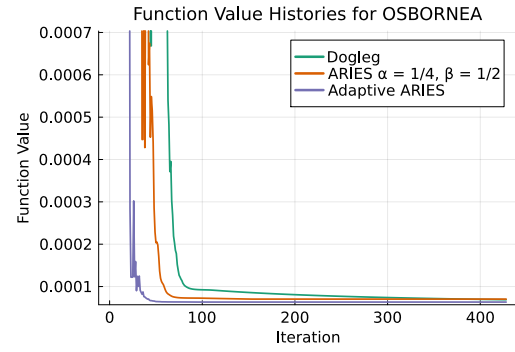
(a) Brownal



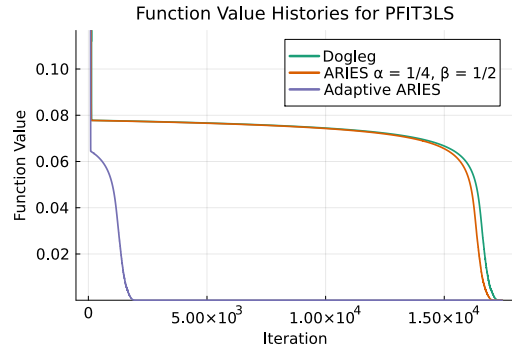
(b) Cyclootls



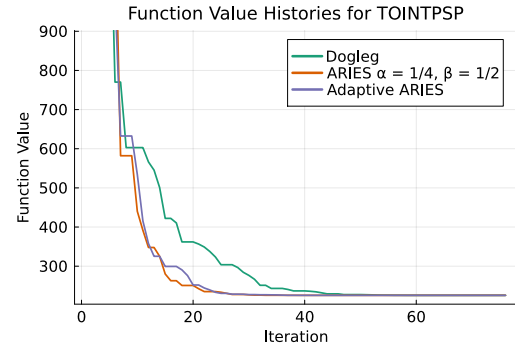
(c) Modbeale



(d) Osbornea



(e) Pfit3ls



(f) Tointpsp

Figure 2: Evolution of function values for selected S2MPJ problems illustrating Aries performance relative to dogleg.

Function	Dogleg	Aries	Adaptive Aries
BROWNAL	0.008	0.007	0.007
CYCLOOCTLS	0.022	0.015	0.017
MODBEALE	0.011	0.011	0.012
OSBORNEA	0.090	0.048	0.037
PFIT3LS	0.587	0.604	0.111
TOINTPSP	0.039	0.030	0.030
Median	0.031	0.023	0.023

Table 1: CPU times (in seconds) for selected S2MPJ problems illustrating Aries performance relative to dogleg.

The iteration-count comparison in Table 2 mirrors the CPU-time trends. Aries requires on average **0.66** of the iterations needed by Dogleg, with a median reduction of 53.5 versus 80 iterations (a 33% decrease). Again, PFIT3LS is the only case where Aries does not improve iteration count; the adaptive variant reduces the iteration count dramatically ($\approx 12\times$ fewer iterations), confirming that the adaptive parameter rule is especially beneficial for difficult problems.

Function	Dogleg	Aries	Adaptive Aries
BROWNAL	76	42	46
CYCLOOCTLS	92	49	54
MODBEALE	84	57	71
OSBORNEA	428	154	97
PFIT3LS	17448	17181	2089
TOINTPSP	76	50	47
Median	80.00	53.50	62.50

Table 2: Iteration counts for selected S2MPJ problems illustrating Aries performance relative to dogleg.

Overall, these results demonstrate that Aries, particularly its adaptive version, offers a consistent improvement in both computational effort and iteration efficiency across a representative set of S2MPJ problems. The observed speed-ups, together with the modest increase in algorithmic complexity, suggest that Aries is a practical replacement for Dogleg in many large-scale trust-region applications.

5.1 Experimental framework

The experimental framework applied uniformly to all subproblem solvers comprises the following components. The SPD matrix B_k is updated via the BFGS scheme [4, 7, 8, 13], with positive-definiteness ensured by a Bunch–Kaufman factorization and additional regularization when the matrix is ill-conditioned. The trust-region radius is adjusted according to the standard strategy [11, 14, 5]: if the actual-to-predicted reduction ratio ρ falls below $c_1 = 1/4$, the radius is contracted; if $\rho > c_2 = 1/2$ and the step is not a Newton step, the radius is expanded; otherwise it remains unchanged, bounded between $0.01\sqrt{n}$ and $10\sqrt{n}$. Steps are accepted whenever $\rho \geq c_0 = 0.1$. The algorithm terminates when the gradient norm drops below 10^{-4} or when the iteration count exceeds 10^5 , with a maximum runtime of one

hour.

6 Conclusions

This paper introduced Aries, a three-segment approximation to the trust-region subproblem. Its additional segment offers flexibility at a small arithmetic cost. Under the assumptions stated above, Aries satisfies the sufficient-decrease condition required by the standard trust-region convergence theory. In the reported S2MPJ experiments, the adaptive rule in (34) gives:

- the largest limiting solve fraction in both profiles, approximately 84.5%;
- the highest fraction of fastest solves in the time profile, approximately 35%; and
- competitive iteration performance relative to dogleg and fixed-parameter Aries.

These results indicate that, while maintaining the same algorithmic simplicity as the classic dogleg method, Aries—especially with the adaptive parameter rule—delivers better average performance than dogleg on the S2MPJ test set. Further experiments with alternative adaptive parameter rules and large-scale problems are warranted.

References

- [1] N. ANDREI, *Nonlinear Conjugate Gradient Methods for Unconstrained Optimization*, Springer International Publishing, Switzerland, 2020.
- [2] R. J. BARALDI AND D. P. KOURI, *Efficient proximal subproblem solvers for a nonsmooth trust-region method*, Computational Optimization and Applications, 90 (2025), p. 193–226.
- [3] J. BEZANSON, A. EDELMAN, S. KARPINSKI, AND V. B. SHAH, *Julia: A fresh approach to numerical computing*, SIAM Review, 59 (2017), pp. 65–98.
- [4] C. G. BROYDEN, *The convergence of a class of double-rank minimization algorithms 1. general considerations*, IMA Journal of Applied Mathematics, 6 (1970), p. 76–90.
- [5] A. R. CONN, N. I. M. GOULD, AND P. L. TOINT, *Trust Region Methods*, Society for Industrial and Applied Mathematics, Jan. 2000.
- [6] E. D. DOLAN AND J. J. MORÉ, *Benchmarking optimization software with performance profiles*, Mathematical Programming, 91 (2002), p. 201–213.
- [7] R. FLETCHER, *A new approach to variable metric algorithms*, The Computer Journal, 13 (1970), p. 317–322.
- [8] D. GOLDFARB, *A family of variable-metric methods derived by variational means*, Mathematics of Computation, 24 (1970), p. 23–26.
- [9] S. GRATTON AND P. L. TOINT, *S2mpj and cutest optimization problems for matlab, python and julia*, Optimization Methods and Software, 40 (2025), p. 871–903.

- [10] D. P. KOURI, *A matrix-free trust-region newton algorithm for convex-constrained optimization*, Optimization Letters, 16 (2021), p. 983–997.
- [11] J. NOCEDAL AND S. J. WRIGHT, *Numerical Optimization*, Springer New York, New York, NY, 2 ed., 2006.
- [12] M. J. D. POWELL, *A hybrid method for nonlinear equations*, Numerical Methods for Nonlinear Algebraic Equations, (1970), pp. 87–161.
- [13] D. F. SHANNO, *Conditioning of quasi-newton methods for function minimization*, Mathematics of Computation, 24 (1970), p. 647–656.
- [14] G. A. SHULTZ, R. B. SCHNABEL, AND R. H. BYRD, *A Family of Trust-Region-Based Algorithms for Unconstrained Minimization with Strong Global Convergence Properties*, SIAM Journal on Numerical Analysis, 22 (1985), p. 47–67.