

SDDP has polynomial iteration complexity in the horizon

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Abstract

Stochastic dual dynamic programming (SDDP) and its variants are run in practice with forward passes that sample the noise at random. Recent iteration-complexity results favour another choice: when the trial points are selected deterministically, for instance along the scenario with the largest gap, the number of iterations is polynomial in the horizon T , whereas every bound we know of for randomly sampled forward passes is exponential in T . We show that this is not intrinsic to random sampling, for a broad class of trajectory-following dynamic programming algorithms, which includes SDDP, stochastic dual dynamic integer programming and stochastic Lipschitz dynamic programming. With random sampling, and with Lipschitz constants, diameters and per-stage loss ranges bounded independently of the stage and of the horizon, the expected number of iterations whose policy is not $T\varepsilon$ -optimal is $O(T\varepsilon^{-(d+1)})$, where d is the state dimension: linear in T , as for deterministic selection rules. The proof needs no particular scenario to be sampled: the gap between the expected cost of the current policy and the current lower bound equals the expected sum of one-step Bellman residuals along the sampled path, and Lipschitz cuts that are tight at the visited states turn the positive residuals into permanent increases of a bounded potential. The results extend to noise without finite support, to backward passes on a sample average approximation, and to stationary and periodic infinite-horizon SDDP with geometric pass lengths.

Keywords: stochastic dual dynamic programming; iteration complexity; multistage stochastic optimisation; random sampling; real-time dynamic programming.

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1 Introduction

Trajectory-following dynamic programming (TFDP) (Forcier and Leclère, 2023) solves multistage stochastic programs with stagewise-independent noise by alternating two passes. A forward pass simulates a trajectory of the policy induced by the current (lower) approximations of the cost-to-go functions; a backward pass adds, at each visited state, a cut, affine or not, that tightens the approximation there. Stochastic dual dynamic programming (SDDP) (Pereira and Pinto, 1991), the most widely used instance, has affine Benders cuts and is run in practice, for instance in SDDP.jl (Dowson and Kapelevich, 2021), with forward passes that sample the noise at random. Almost sure convergence of these methods is classical (Philpott and Guan, 2008; Shapiro, 2011; Girardeau et al., 2015; Zou et al., 2019). See also the survey of Lan and Shapiro (2024) and the review of Füllner and Rebennack (2025).

Known bounds. For deterministic selection rules of the trial points, such as the “problem child” rule that sends the forward pass to the scenario with the largest current gap, the number of iterations is polynomial in the horizon T for a fixed state dimension d (Lan, 2022a; Zhang and Sun, 2022;

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setting	expected number of bad iterations	reference
general oracle, volume argument	$(d+1)\bar{\ell}(T+1)(4LD)^d/\varepsilon^{d+1}$	Corollary 3.7
general oracle, covering (no convexity)	$Z\bar{\ell}(T+1)(1+8LD/\varepsilon)^d/\varepsilon$	Corollary 3.10
Zhang–Sun, Algorithm 3	$\bar{\ell}(T+1)(1+8LD/\varepsilon)^d/\varepsilon$	Theorem 4.3
SDDiP, binary states	$2^{n-1}\bar{\ell}(T+1)/\varepsilon$	Proposition 4.4
discounted, any T ; infinite horizon, geometric length (accuracy ε)	$2(d+1)\bar{\ell}(4\hat{L}D)^d/((1-\beta)^{d+2}\varepsilon^{d+1})$	Corollaries 3.9 and 6.6

Table 1: An iteration k is *bad* if $G^k > T\varepsilon$ ($G^k > \varepsilon$ in the discounted row). For Zhang and Sun’s algorithm, the row bounds the index of the first good iteration. Z , n , β and $\hat{L}$ are defined with the cited results.

[Forcier and Leclère, 2023](#); [Ju and Lan, 2023](#)). For randomly sampled forward passes, every bound we know of is exponential in T ([Lan, 2022b](#); [Zhang and Sun, 2022](#); [Forcier and Leclère, 2023](#); [Ju and Lan, 2023](#)), including the dimension-free bounds of [Barros and Guigues \(2026\)](#) under strong convexity. The reason is common to all these proofs. They count *saturated* points, where the true gap $V_t - \underline{V}_t^k$ is certified small, an argument introduced by [Lan \(2022a\)](#) and extended to nonconvex problems by [Zhang and Sun \(2022\)](#) through Lipschitz regularisation. But saturation at stage t follows from saturation at stage $t+1$ only at the point reached by the same sampled path. Each proof therefore bounds from below the probability that the sampled path follows a given chain of scenarios, for instance by $\prod_t 1/N_t$ under uniform sampling on N_t atoms, which is exponentially small in T . [Zhang and Sun](#) ask, in their Remark 3, whether a bound polynomial in T exists in general for the random-sampling version.

Main result. We prove bounds polynomial in T for TFDP-type algorithms with randomly sampled forward passes. Let π^k be the policy of iteration k , greedy for the approximations of iteration $k-1$, and $G^k := V_0^{\pi^k}(x_0) - \underline{V}_0^{k-1}(x_0)$ the gap between its expected cost and the lower bound. Assume that, at every stage, the set of states has dimension at most d and diameter at most D , that the cuts are L -Lipschitz on it, and that the initial lower bounds are within $\bar{\ell}$ per remaining stage of the cost-to-go functions (e.g. when the stage losses lie in an interval of width $\bar{\ell}$, [Section 3.3](#)). Then the expected number of iterations with $G^k > T\varepsilon$, an error of ε per stage, is $O(T\varepsilon^{-(d+1)})$, with a constant that depends only on d , D , L and $\bar{\ell}$ ([Corollary 3.7](#)). In the discounted case this number does not depend on the horizon ([Corollary 3.9](#)). All constants are explicit ([Table 1](#)).

For the same accuracy, and with each state set in a ball of diameter D (a convention under which our bound also holds), the deterministic problem-child rule needs at most $T(1+2LD/\varepsilon)^d$ iterations ([Zhang and Sun, 2022](#)), so random sampling costs a factor of about $(d+1)2^d\bar{\ell}/\varepsilon$, independent of T ([Section 7.1](#)). Linearity in T needs Lipschitz constants bounded independently of the stage and of the horizon, as in the inventory model of [Example 4.2](#); under extended relatively complete recourse, the Lipschitz constants of Benders cuts are only known to grow like $T-t$ ([Proposition 4.1](#)). As for all random-sampling results, G^k is not observed: the bounds count bad iterations and do not provide a stopping test ([Remark 2.1](#)).

Proof outline. The proof does not use saturation. It follows the analysis of real-time dynamic programming (RTDP) ([Barto et al., 1995](#)) in reinforcement learning: the SDDP forward pass is exactly the sampling scheme of RTDP, which follows greedy trajectories, drawn from the true transition law, of a policy built on an optimistic value estimate (here a lower bound on costs). Let $\underline{V}_t^k$ be the approximation of the cost-to-go function V_t at time t after k iterations, $\mathcal{B}_t$ the Bellman

operator between times t and $t + 1$, $(\mathbf{x}_t^k)_{0 \leq t \leq T}$, with $\mathbf{x}_0^k = x_0$, the states visited by the forward pass of iteration k , and $\mathcal{A}_{k-1}$ the history of the first $k - 1$ iterations. Besides the gap, two quantities are tracked: the one-step Bellman residual $r_t^k := \mathcal{B}_t(\underline{V}_{t+1}^{k-1}) - \underline{V}_t^{k-1}$, which is local in time, and the volume $\Phi^k := \sum_{t < T} \int (\underline{V}_t^k - \underline{V}_t^0) d\mu_t$ gained by the approximations, for a reference probability measure μ_t on the states at time t (the normalised Lebesgue measure when the state set is convex). With exact cuts and an exact forward pass, three facts suffice.

- (i) *Gap identity* (Lemma 3.2), as in Efroni et al. (2019, Lemma 2) for RTDP. The gap equals the expected sum of the residuals along the path that the algorithm samples, so no particular path needs to be sampled:

$$G^k = \mathbb{E} \left[\sum_{t < T} r_t^k(\mathbf{x}_t^k) \mid \mathcal{A}_{k-1} \right]. \quad (1)$$

- (ii) *Lift* (Lemma 3.3). The backward pass lifts $\underline{V}_t^k$ by its residual at the trial point; the approximations being Lipschitz, this pointwise lift is a lift on a ball, hence of the integral:

$$\Phi^k - \Phi^{k-1} \geq \Psi_T \left(\sum_{t < T} r_t^k(\mathbf{x}_t^k) \right) \quad (2)$$

for an explicit convex function Ψ_T .

- (iii) *Bounded potential*. The approximations are nondecreasing and stay below the value functions, so $0 \leq \Phi^k \leq \sum_{t < T} M_t$ with $M_t \leq \sup(V_t - \underline{V}_t^0)$.

Taking expectations in (ii), using (i) and Jensen's inequality, and summing over k with (iii) gives

$$\mathbb{E} \sum_{k \geq 1} \Psi_T(G^k) \leq \sum_{t < T} M_t \quad (3)$$

(Theorem 3.5), from which all the rates follow. In metric state spaces, Domingues et al. (2021, App. F) and Sinclair et al. (2020, Lemma 7.3) already accumulate Lipschitz value updates of this kind over a covering and over adaptive partitions. We transfer this residual accounting to a general cut oracle: approximate tightness supplies the local lift, while a fixed upper bound on the cuts bounds the potential. To our knowledge, no earlier paper applies the RTDP analysis to SDDP or to cutting-plane methods.

Noise without finite support. Two results seem new. When the cuts are computed on a fixed SAA and the forward passes are drawn from the true law, a common practice to which earlier convergence proofs do not apply (Shapiro, 2011), the bounds hold for the true problem up to one-sided per-stage biases; for an i.i.d. SAA of size N these are at most of order $N^{-1/(d+2)}$ up to a logarithm, a rate governed by the state dimension rather than the noise dimension. Cuts computed from a fresh sample at each backward step are valid only in expectation; stochastic decomposition keeps such cuts usable by scaling them down deterministically (Higle and Sen, 1991; Sen and Zhou, 2014). Without a correction they can make SDDP stall on a suboptimal policy; lowering each one by a Hoeffding radius makes all of them valid with high probability and restores the bounds, which extends to the multistage setting the confidence lower bounds of Glynn and Infanger (2013) for two-stage sampled cutting planes.

Organisation. Section 2 sets out the problem, the algorithms and the cut oracle, whose cuts need only be Lipschitz, approximately tight at the trial point and below a fixed *ceiling*, on state sets that may be nonconvex, mixed-integer or finite. Section 3 proves the master theorem and derives explicit rates. Section 4 applies it to Benders cuts, to the generalised conjugacy cuts of Zhang and Sun, and to stochastic dual dynamic integer programming (SDDiP) on binary states.

Section 5 treats noise without finite support: valid inexact cuts, backward passes on a sample average approximation (SAA), and sampled cuts. Section 6 extends the analysis to stationary and periodic infinite-horizon problems and gives, for SDDP under extended relatively complete recourse, bounds polynomial in the effective horizon $1/(1 - \beta)$. Section 7 compares the results with the SDDP and reinforcement-learning literature, and Appendix E collects the notation.

2 Setting

This section fixes the problem, the algorithms and the accuracy measure, the gap G^k , and states the two assumptions of the analysis, on the cut oracle (Section 2.3) and on the state sets (Section 2.4).

2.1 Problem

We consider a multistage stochastic program with stagewise-independent noise. Stage $t \in [T] := \{t \in \mathbb{N} : 1 \leq t \leq T\}$ starts at time $t - 1$ in a state $x_{t-1} \in X_{t-1}$, observes the noise ξ_t , with law $\mathbb{P}_t$ on a measurable space Ξ_t and independent across stages, chooses the next state $x_t \in \mathcal{X}_t(x_{t-1}, \xi_t) \subset X_t \subset \mathbb{R}^{n_t}$, and pays the loss $\ell_t(x_{t-1}, x_t, \xi_t)$; the initial state is fixed, $X_0 = \{x_0\}$. We write $\xi_{[t]} := (\xi_s)_{s \in [t]}$ for the noise history up to stage t . The loss ℓ_t is Borel and real-valued (infinite losses are encoded in the constraint set $\mathcal{X}_t$). For $t < T$, a bounded Borel function W on X_{t+1} and a probability law $\mathbb{P}$ on Ξ_{t+1} , let

$$\begin{aligned} \dot{\mathcal{B}}_t(W; x, \xi) &:= \inf_{y \in \mathcal{X}_{t+1}(x, \xi)} \ell_{t+1}(x, y, \xi) + W(y), \\ \mathcal{B}_t^{\mathbb{P}}(W)(x) &:= \int \dot{\mathcal{B}}_t(W; x, \xi) \mathbb{P}(d\xi), \quad x \in X_t. \end{aligned} \tag{4}$$

We write $\mathcal{B}_t := \mathcal{B}_t^{\mathbb{P}_{t+1}}$ for the Bellman operator of the true law, and set $V_T := 0$ and $V_t := \mathcal{B}_t(V_{t+1})$ for $t < T$ (backward in t). The operator $\mathcal{B}_t$ is monotone ($W \leq W'$ on X_{t+1} implies $\mathcal{B}_t(W) \leq \mathcal{B}_t(W')$ on X_t) and commutes with constants. The following mild conditions make the stage problems and the cost of a policy well defined.

Assumption A (Standing assumptions). For every $t \in [T]$:

- (S1) X_{t-1} is compact and, for all $x \in X_{t-1}$ and $\xi \in \Xi_t$, $\emptyset \neq \mathcal{X}_t(x, \xi) \subset X_t$ (forward invariance); X_T need not be compact ($V_T = 0$ on it).
- (S2) (Integrability.) $\sup_{x \in X_{t-1}} \int |\inf_{y \in \mathcal{X}_t(x, \xi)} \ell_t(x, y, \xi)| \mathbb{P}_t(d\xi) < \infty$.
- (S3) (Measurability.) Measurable selections of the stage problems exist, the approximations of Section 2.2 are Borel functions of finitely many random parameters (for instance cut coefficients), and all oracles are Borel functions of their inputs.

Condition (S2) implies that $\mathcal{B}_t(0)$ is bounded on X_t for $t < T$, hence $|\mathcal{B}_t(W)| \leq \sup_{X_t} |\mathcal{B}_t(0)| + \sup |W|$. Condition (S3) is automatic for finitely supported noise and deterministic oracles with a tie-breaking rule; Appendix A.4 gives the precise statement and a sufficient condition (a normal integrand and approximations continuous in the state).

2.2 Randomised TFDP-type algorithms

The algorithm keeps approximations $\underline{V}_t^k : X_t \rightarrow \mathbb{R}$ of the value functions, $0 \leq t \leq T$, with $\underline{V}_T^k := 0$, where k counts the completed iterations: quantities carrying the superscript k are produced during iteration k , which uses the approximations of index $k - 1$. The approximations need not lie below V_t (Section 2.3).

A *forward operator* $\mathcal{F}_{t-1}^k$ maps a bounded Borel function W on X_t to a selection $\mathcal{F}_{t-1}^k(W)(x, \xi) \in \mathcal{X}_t(x, \xi)$ that is greedy for W up to a *forward error* $\gamma_t^{F,k} \geq 0$: for all x and ξ with $\dot{\mathcal{B}}_{t-1}(W; x, \xi) > -\infty$,

$$\ell_t(x, \mathcal{F}_{t-1}^k(W)(x, \xi), \xi) + W(\mathcal{F}_{t-1}^k(W)(x, \xi)) \leq \dot{\mathcal{B}}_{t-1}(W; x, \xi) + \gamma_t^{F,k}. \quad (5)$$

We write $\gamma^F = 0$ for exact greedy selections ($\gamma_t^{F,k} = 0$ for all k and t). The policy π^k of iteration k is greedy for the approximations of the previous iteration: at stage t , in state x and under noise ξ , it moves to $\mathcal{F}_{t-1}^k(\underline{V}_t^{k-1})(x, \xi)$.

Let $\mathcal{A}_k$ be the σ -field generated by everything drawn during the first k iterations, including any internal randomness of the oracles, which must be independent of the noise of the current forward pass. The forward operators and the forward errors of iteration k are $\mathcal{A}_{k-1}$ -measurable (Appendix A.4), so π^k is known before iteration k starts. Iteration $k \geq 1$ runs two passes.

- *Forward pass.* Draw $(\xi_t^k)_{t \in [T]}$ independently, with laws $(\mathbb{P}_t)_{t \in [T]}$ and independently of $\mathcal{A}_{k-1}$, and simulate π^k from $\mathbf{x}_0^k := x_0$, with $\mathbf{x}_t^k := \mathcal{F}_{t-1}^k(\underline{V}_t^{k-1})(\mathbf{x}_{t-1}^k, \xi_t^k)$ for $t \in [T]$.
- *Backward pass.* For t from $T-1$ down to 0, an oracle returns a function $f_t^k : X_t \rightarrow \mathbb{R}$, the *cut*, at the trial point $\mathbf{x}_t^k$, possibly using the already updated $\underline{V}_{t+1}^k$. Then $\underline{V}_t^k := \max(\underline{V}_t^{k-1}, f_t^k)$, possibly with further cuts, see (O4).

SDDP is the case of finitely supported noise, exact greedy passes and Benders cuts. The general case is TFDP with random node selection from the law of ξ_t , whose upper approximations we do not use.

Cost, gap and residuals. Since the forward pass of iteration k is a trajectory of π^k drawn independently of $\mathcal{A}_{k-1}$, the expected cost of π^k and the *gap* between this cost and the lower bound are

$$V_0^{\pi^k}(x_0) := \mathbb{E} \left[\sum_{t \in [T]} \ell_t(\mathbf{x}_{t-1}^k, \mathbf{x}_t^k, \xi_t^k) \mid \mathcal{A}_{k-1} \right], \quad G^k := V_0^{\pi^k}(x_0) - \underline{V}_0^{k-1}(x_0). \quad (6)$$

The *residuals* r_t^k measure how far the approximations are from satisfying the Bellman equation; we also use their sum along the sampled path:

$$r_t^k := \mathcal{B}_t(\underline{V}_{t+1}^{k-1}) - \underline{V}_t^{k-1} \quad \text{on } X_t \ (t < T), \quad R^k := \sum_{t < T} r_t^k(\mathbf{x}_t^k). \quad (7)$$

Residuals are measured with the true operator $\mathcal{B}_t$, whatever the oracle computes. They may be negative, since the sub-Bellman inequality $\underline{V}_t^{k-1} \leq \mathcal{B}_t(\underline{V}_{t+1}^{k-1})$ is not assumed; they are nonnegative under the conditions of Lemma 2.2(a).

What the gap measures. All results are stated in terms of the gap G^k defined in (6). It cannot be observed, but it controls both the policy and the lower bound. Let $\zeta_0 := (U_0(x_0) - V_0(x_0))_+$, where U_0 is the ceiling of Section 2.3; $\zeta_0 = 0$ in the standard case $U = V$. By Lemma 3.2 below, for every $k \geq 1$,

$$G^k \geq V_0(x_0) - \underline{V}_0^{k-1}(x_0) \quad \text{and} \quad G^k \geq V_0^{\pi^k}(x_0) - V_0(x_0) - \zeta_0. \quad (8)$$

So, when $\zeta_0 = 0$, $G^k \leq \theta$ means that the whole policy π^k (not only its first decision) is θ -optimal and that the lower bound $\underline{V}_0^{k-1}(x_0)$ is θ -tight.

Remark 2.1 (Comparison of accuracy notions). With $\zeta_0 = 0$, $G^k \leq \theta$ implies the guarantees of Lan (2022a, eqs. (5.18)–(5.19)) (a θ -optimal first-stage solution, since $V_0^{\pi^k}(x_0) \geq \mathbb{E}[\ell_1(x_0, \mathbf{x}_1^k, \xi_1^k) + V_1(\mathbf{x}_1^k)]$), of Zhang and Sun (2022, Def. 1) (a θ -optimal root-node solution) and of Forcier and Leclère (2023, Thms. 4.2 and 4.11) (a θ -solution and a θ -tight lower bound). These works bound the first index

at which their guarantee holds. That index is at most the first index with $G^k \leq \theta$, hence at most $1 + \#\{k \geq 1 : G^k > \theta\}$, which our bounds control. Such bounds are of uniform-PAC type (Dann et al., 2017). As in all earlier results on random sampling, identifying a good iteration requires Monte Carlo estimates of $V_0^{\pi^k}(x_0)$, unless one returns a randomly selected iterate (Corollary 3.6(d)).

2.3 The cut oracle

In the standard TFDP case, each cut is *valid*, that is, below V_t , and *tight* at the trial point. Assumption B weakens both: tightness is required only up to an error γ , and with respect to the approximation of the previous iteration (O2); validity is replaced by a *ceiling* (O3), a function U_t above all cuts. A ceiling other than V_t is needed when the forward pass is greedy for one problem while the cuts lie below the value functions of another: a regularised and the original problem in Section 4.2, the true law and an SAA in Section 5.3.

Assumption B (Lipschitz cut oracle). For each $t < T$ there are a metric dist_t on X_t , possibly extended (with values in $[0, +\infty]$), a constant $L_t \geq 0$ and a *ceiling* $U_t : X_t \rightarrow (-\infty, +\infty]$ such that, for all $k \geq 1$:

- (O1) (Lipschitz) $\underline{V}_t^0$ and every cut of index t are Borel and L_t -Lipschitz for dist_t , and $\underline{V}_t^0$ is deterministic and bounded.
- (O2) (tightness at the trial point, up to γ) At $x = \mathbf{x}_t^k$, $f_t^k(x) \geq \mathcal{B}_t(\underline{V}_{t+1}^{k-1})(x) - \gamma_t^k$, where the error $\gamma_t^k \geq 0$ is integrable and $\mathcal{A}_k$ -measurable.
- (O3) (ceiling) $\underline{V}_t^0 \leq U_t$ and every cut of index t is $\leq U_t$ on X_t . Moreover the *potential bound* $M_t := \sup_{X_t} (\bar{U}_t - \underline{V}_t^0)$ is finite, where $\bar{U}_t(x) := \inf_{z \in X_t} \{U_t(z) + L_t \text{dist}_t(x, z)\}$, with $L_t \cdot \infty := \infty$.
- (O4) (monotone update) $\underline{V}_t^k = \max(\{\underline{V}_t^{k-1}, f_t^k\} \cup \{f'_i\}_{i \in [m]})$ for some $m \geq 0$, where the additional cuts f'_i , if any, satisfy (O1) and (O3).

The function $\bar{U}_t$, the *Lipschitz envelope* of the ceiling, is the largest L_t -Lipschitz function below U_t ; if dist_t is finite and $U_t(z_0) < \infty$, then $M_t \leq U_t(z_0) + L_t \text{diam}_{\text{dist}_t}(X_t) - \inf \underline{V}_t^0$, and if $U_t = V_t$ is L_t -Lipschitz, then $M_t = \sup(V_t - \underline{V}_t^0)$. The usual validity and tightness conditions imply Assumption B:

Lemma 2.2 (Sufficient conditions). Assume (S1)–(S2), and let each $\underline{V}_t^0$ be bounded.

- (a) (Strong validity implies (O3).) Assume $\underline{V}_t^0 \leq V_t$ for all t , and that every cut created at index $t < T$ during iteration k satisfies $f \leq \mathcal{B}_t(\underline{V}_{t+1}^k)$ on X_t ; this is the validity notion of TFDP (Forcier and Leclère, 2023, Assumption 4(ii)) and of stochastic Lipschitz dynamic programming (SLDP) (Ahmed et al., 2022, eq. (20)). Then every cut is below V_t on X_t , so (O3) holds with $U_t = V_t$. If moreover $\underline{V}_t^0 \leq \mathcal{B}_t(\underline{V}_{t+1}^0)$, then $\underline{V}_t^k \leq \mathcal{B}_t(\underline{V}_{t+1}^k)$ on X_t for all $k \geq 0$, so that $r_t^k \geq 0$ for all $k \geq 1$.
- (b) (Tightness for the updated approximation implies (O2).) If $f_t^k(x) \geq \mathcal{B}_t(\underline{V}_{t+1}^k)(x) - \gamma_t^k$ at $x = \mathbf{x}_t^k$ and $\underline{V}_{t+1}^k \geq \underline{V}_{t+1}^{k-1}$, then (O2) holds.

The proof is in Appendix A.1. The following are *not* assumed: convexity of V_t , of ℓ_t or of the graph of $\mathcal{X}_t(\cdot, \xi)$; affine cuts; Lipschitz continuity of V_t . Convexity only serves, through the oracle, to guarantee that good cuts exist.

2.4 Conditions on the state sets

The backward pass raises an approximation at a single trial point; the main theorem turns this into progress on the whole state set in two ways, each with its condition on X_t : through the volume under the approximation, or through its values on a finite net. The volume version (Lemma 3.3) uses

the following *volume-growth* condition: there are a Borel probability measure μ_t on X_t , constants $\kappa_t \in (0, 1]$ and $D_t > 0$, and an integer $d_t \geq 0$ such that, with the convention $0^0 = 1$,

$$\mu_t(\{y \in X_t : \text{dist}_t(x, y) \leq s\}) \geq \kappa_t \min\{1, (s/D_t)^{d_t}\} \quad \forall x \in X_t, s \geq 0. \quad (\text{VG}_t)$$

We always take $d_0 = 0$ and $\kappa_0 = 1$ (recall $X_0 = \{x_0\}$), and we fix an exponent $d \geq \max_{t < T} d_t$.

Lemma 2.3 (Cases of (VG_t)). (a) *Convex sets.* If X_t is convex and compact, dist_t is given by a norm, $d_t := \dim \text{aff } X_t$, $D_t \geq \text{diam } X_t$ and μ_t is the normalised Lebesgue measure of $\text{aff } X_t$ on X_t , then (VG_t) holds with $\kappa_t = 1$.

(b) *Finite unions of convex sets (mixed-integer states).* If $X_t = \bigcup_{j \in I_t} K^j$ with I_t finite and each K^j convex and compact, $\mu_t := |I_t|^{-1} \sum_j \mu_{K^j}$ with μ_{K^j} as in (a), $d_t \geq \max_j \dim \text{aff } K^j$ and $D_t \geq \max_j \text{diam } K^j$, then (VG_t) holds with $\kappa_t = 1/|I_t|$ for dist_t given by a norm. If the K^j are pairwise disjoint, it also holds for the extended metric equal to $\|x - y\|$ when x, y lie in the same K^j and $+\infty$ otherwise; for that metric, (O1) only asks for Lipschitz continuity on each piece. For instance, $X_t = \bigcup_{z \in Z_t} \{z\} \times K_z$ with Z_t finite gives $\kappa_t = 1/|Z_t|$.

(c) *Finite sets.* If X_t is finite and μ_t is uniform, (VG_t) holds with $\kappa_t = 1/|X_t|$ and $d_t = 0$, for any dist_t .

The proof is in [Appendix A.1](#). Condition (VG_t) can fail for compact sets with outward cusps: at the origin of $\{(u, v) : 0 \leq u \leq 1, |v| \leq u^2\}$, the Lebesgue measure of a ball of radius s is of order s^3 , not s^2 . The covering version of the main theorem needs no such condition, only finite nets. A set $C \subset X_t$ is a σ -net for dist_t if every $x \in X_t$ has a point $c \in C$ with $\text{dist}_t(x, c) \leq \sigma$. By volume packing ([Lemma A.1.1](#)), a compact subset of a d_t -dimensional affine space with diameter at most D_t has σ -nets of size $(1 + 2D_t/\sigma)^{d_t}$, and a union of $|Z_t|$ such fibres has nets of size $|Z_t|(1 + 2D_t/\sigma)^{d_t}$.

The standard case. A first reading may assume the standard case: SDDP with an exact forward pass ($\gamma^F = 0$), cuts that are valid ($U = V$) and exact at the trial point ($\gamma = 0$), convex compact state sets and dist_t given by a norm. Then $\kappa_t = 1$ and $d_t = \dim \text{aff } X_t$ ([Lemma 2.3\(a\)](#)). The other cases serve later results: extended metrics and finite sets for mixed-integer and binary states, and finite nets for nonconvex state sets.

3 Main results

This section proves the master theorem ([Theorem 3.5](#)) along the three steps outlined in the introduction, then turns it into anytime guarantees and explicit rates.

3.1 Gap identity and lift

The errors of an iteration add up to

$$\Gamma_k := \sum_{t < T} \gamma_t^k, \quad \Gamma_k^F := \sum_{t \in [T]} \gamma_t^{F,k}, \quad \bar{\Gamma}_k := \Gamma_k^F + \mathbb{E}[\Gamma_k \mid \mathcal{A}_{k-1}], \quad (9)$$

and, for deterministic errors $\gamma_t^k = \gamma_t$ and $\gamma_t^{F,k} = \gamma_t^F$, we write $\Gamma := \sum_{t < T} \gamma_t + \sum_{t \in [T]} \gamma_t^F$.

Lemma 3.1 (Bounds). Under [Assumptions A](#) and [B](#), we have, for all $k \geq 1$ and $t < T$:

- (i) $\underline{V}_t^k$ is Borel and L_t -Lipschitz for dist_t , and $\underline{V}_t^0 \leq \underline{V}_t^{k-1} \leq \underline{V}_t^k \leq \bar{U}_t$ on X_t , where $\bar{U}_t$ is the Lipschitz envelope of the ceiling [\(O3\)](#);
- (ii) the residual r_t^k of [\(7\)](#) is bounded on X_t , uniformly in k ;

(iii) $V_0^{\pi^k}(x_0) \geq V_0(x_0)$.

Proof. (i) A finite maximum of Borel L_t -Lipschitz functions is Borel and L_t -Lipschitz. If f is L_t -Lipschitz and $f \leq U_t$, then $f(x) \leq f(z) + L_t \text{dist}_t(x, z) \leq U_t(z) + L_t \text{dist}_t(x, z)$ for all z , so $f \leq \bar{U}_t$. This applies to $\underline{V}_t^0$ and to every cut, and (O4) gives the monotonicity. (ii) $\underline{V}_t^0 \leq \bar{U}_t \leq M_t + \sup \underline{V}_t^0$ by the definition of the potential bound M_t in (O3), so $\bar{U}_t$ is bounded; then use $|\mathcal{B}_t(W)| \leq \sup |\mathcal{B}_t(0)| + \sup |W|$ with $W = \underline{V}_{t+1}^{k-1}$. (iii) The realised costs of π^k are integrable given $\mathcal{A}_{k-1}$ (proof of Lemma 3.2), and a policy defined by measurable selections with integrable costs satisfies $\mathbb{E}[\sum_{s>t} \ell_s \mid \mathbf{x}_t] \geq V_t(\mathbf{x}_t)$, by backward induction and stagewise independence. $\square$

Lemma 3.2 (Gap identity). *Under Assumption A, let $\underline{V}_t^{k-1}$, $t \leq T$, be any bounded approximations as in the measurability condition (S3), with $\underline{V}_T^{k-1} = 0$. Then the gap G^k of (6) and the sum R^k of the residuals along the sampled path (7) satisfy*

$$G^k = \mathbb{E}[R^k \mid \mathcal{A}_{k-1}] + \mathbb{E}\left[\sum_{t \in [T]} e_t^k \mid \mathcal{A}_{k-1}\right], \quad e_t^k \in [0, \gamma_t^{F,k}], \quad (10)$$

where $e_t^k := \ell_t(\mathbf{x}_{t-1}, \mathbf{x}_t, \boldsymbol{\xi}_t) + \underline{V}_t^{k-1}(\mathbf{x}_t) - \dot{\mathcal{B}}_{t-1}(\underline{V}_t^{k-1}; \mathbf{x}_{t-1}, \boldsymbol{\xi}_t)$ is the error of the forward step, with $\dot{\mathcal{B}}_{t-1}$ the value of the stage problem (4), and $(\mathbf{x}_t, \boldsymbol{\xi}_t) = (\mathbf{x}_t^k, \boldsymbol{\xi}_t^k)$. In particular $G^k \leq \mathbb{E}[R^k \mid \mathcal{A}_{k-1}] + \Gamma_k^F$, with equality for the exact greedy pass. Under Assumption B, the gap inequality (8) holds.

Proof. Let $\mathcal{H}_{t-1} := \mathcal{A}_{k-1} \vee \sigma(\boldsymbol{\xi}_{[t-1]}^k)$. Then $\mathbf{x}_{t-1}$ is $\mathcal{H}_{t-1}$ -measurable and lies in X_{t-1} by (S1), and $\boldsymbol{\xi}_t$ is independent of $\mathcal{H}_{t-1}$ with law $\mathbb{P}_t$. The substitution (“freezing”) property of independent variables (Kallenberg, 2002) states that if g is jointly measurable, Y is $\mathcal{H}$ -measurable, $\boldsymbol{\xi}$ is independent of $\mathcal{H}$ and $\mathbb{E}[g(Y, \boldsymbol{\xi})] < \infty$, then $\mathbb{E}[g(Y, \boldsymbol{\xi}) \mid \mathcal{H}] = \bar{g}(Y)$ with $\bar{g}(y) := \mathbb{E}g(y, \boldsymbol{\xi})$. It gives $\mathbb{E}[\dot{\mathcal{B}}_{t-1}(\underline{V}_t^{k-1}; \mathbf{x}_{t-1}, \boldsymbol{\xi}_t) \mid \mathcal{H}_{t-1}] = \mathcal{B}_{t-1}(\underline{V}_t^{k-1})(\mathbf{x}_{t-1}) = \underline{V}_{t-1}^{k-1}(\mathbf{x}_{t-1}) + r_{t-1}^k(\mathbf{x}_{t-1})$, so

$$\mathbb{E}[\ell_t(\mathbf{x}_{t-1}, \mathbf{x}_t, \boldsymbol{\xi}_t) + \underline{V}_t^{k-1}(\mathbf{x}_t) \mid \mathcal{H}_{t-1}] = \underline{V}_{t-1}^{k-1}(\mathbf{x}_{t-1}) + r_{t-1}^k(\mathbf{x}_{t-1}) + \mathbb{E}[e_t^k \mid \mathcal{H}_{t-1}]. \quad (11)$$

Take $\mathbb{E}[\cdot \mid \mathcal{A}_{k-1}]$ and sum over $t \in [T]$. The terms in $\underline{V}^{k-1}$ telescope to $-\underline{V}_0^{k-1}(x_0)$, because $\underline{V}_T^{k-1} = 0$ and x_0 is deterministic. The bounds on e_t^k are the forward-error condition (5) and the definition of $\dot{\mathcal{B}}_{t-1}$. All terms are integrable given $\mathcal{A}_{k-1}$: the realised cost satisfies $\inf_y \ell_t(\mathbf{x}_{t-1}, y, \boldsymbol{\xi}_t) \leq \ell_t(\mathbf{x}_{t-1}, \mathbf{x}_t, \boldsymbol{\xi}_t) \leq \inf_y \ell_t(\mathbf{x}_{t-1}, y, \boldsymbol{\xi}_t) + 2 \sup |\underline{V}_t^{k-1}| + \gamma_t^{F,k}$, and $\mathbb{E}[|\inf_y \ell_t(\mathbf{x}_{t-1}, y, \boldsymbol{\xi}_t)| \mid \mathcal{H}_{t-1}]$ is at most $\sup_{x \in X_{t-1}} \int |\inf_y \ell_t(x, y, \boldsymbol{\xi})| \mathbb{P}_t(d\boldsymbol{\xi})$, which is finite by the integrability condition (S2), by the freezing property. For the gap inequality (8): $G^k \geq V_0^{\pi^k}(x_0) - \bar{U}_0(x_0)$ and $\bar{U}_0(x_0) = U_0(x_0)$ because $X_0 = \{x_0\}$, so $G^k \geq V_0^{\pi^k}(x_0) - V_0(x_0) - \zeta_0$ with ζ_0 as in Section 2.2; and $V_0^{\pi^k}(x_0) \geq V_0(x_0)$ by Lemma 3.1(iii). $\square$

For finite state spaces and an exact greedy pass, this lemma is Efroni et al. (2019, Lemma 2) for RTDP; the proof is their telescoping argument, written for general noise, an inexact forward pass and arbitrary approximations. The identity involves the forward law only: it holds whatever the backward pass computes (sampled cuts, cuts for another law, inexact cuts). What the backward pass must do is lift the approximation at the trial point by an amount comparable to the residual there.

Lemma 3.3 (A Lipschitz lift lifts a volume). *Assume the volume-growth condition (VG_t). Let $V, W : X_t \rightarrow \mathbb{R}$ be Borel and L_t -Lipschitz for dist_t , with $W \geq V$, and let $x \in X_t$ and $s \geq 0$ satisfy $W(x) \geq V(x) + s$. Then*

$$\int (W - V) d\mu_t \geq \kappa_t \varphi_t(s), \quad \text{where} \quad \varphi_t(s) := \frac{s}{2} \min \left\{ 1, \left(\frac{s}{4L_t D_t} \right)^{d_t} \right\}, \quad (12)$$

with $s/(4L_t D_t) := \infty$ if $L_t = 0$. If $d_t = 0$, the bound improves to $\kappa_t s$.

Proof. If $\text{dist}_t(x, y) \leq u$, then $W(y) - V(y) \geq W(x) - V(x) - 2L_t u \geq s - 2L_t u$, and $W - V \geq 0$ everywhere. So $\int (W - V) d\mu_t \geq (s - 2L_t u)_+ \kappa_t \min\{1, (u/D_t)^{d_t}\}$ for every $u \geq 0$. Take $u = D_t$ if $s \geq 4L_t D_t$ and $u = s/(4L_t)$ otherwise; if $d_t = 0$, take $u = 0$. $\square$

At iteration k , tightness (O2) and the monotone update (O4) give, at $x = \mathbf{x}_t^k$, $V_t^k(x) \geq \underline{V}_t^{k-1}(x) + (r_t^k(x) - \gamma_t^k)_+$, so Lemma 3.3 applies at every index $t < T$ of every backward pass with $V = \underline{V}_t^{k-1}$, $W = \underline{V}_t^k$ and $s = (r_t^k(x) - \gamma_t^k)_+$.

The aggregated lift function. Let $\rho_t := 4L_t D_t$ and let $\hat{\psi}_t$ be the convex “Huber-type” minorant of φ_t with exponent $d \geq \max_{t < T} d_t$:

$$\hat{\psi}_t(s) := \begin{cases} \frac{s^{d+1}}{2(d+1)\rho_t^d}, & 0 \leq s \leq \rho_t, \\ \frac{\rho_t}{2(d+1)} + \frac{s - \rho_t}{2}, & s \geq \rho_t. \end{cases} \quad (13)$$

When $\rho_t = 0$ (that is, $L_t = 0$, for instance at the root), only the second branch applies: $\hat{\psi}_t(s) = s/2$. For every $\rho_t \geq 0$, $\hat{\psi}_t \leq \varphi_t$ because $d_t \leq d$. Set $\psi_t := \kappa_t \hat{\psi}_t$ and let $\Psi_T := \square_{t < T} \psi_t$ be their infimal convolution, $\Psi_T(g) = \min\{\sum_{t < T} \psi_t(s_t) : s \geq 0, \sum_{t < T} s_t = g\}$. It is convex, finite and nondecreasing on $[0, \infty)$, with $\Psi_T(0) = 0$, and we extend it to $\mathbb{R}$ by $\Psi_T(u) := \Psi_T(u_+)$, which keeps it convex and nondecreasing. This *aggregated lift function* $\Psi_T(g)$ is the smallest total lift that these minorants guarantee when the residuals along a path, net of the cut errors, add up to g . For small g and stage-uniform constants, $\Psi_T(g) \approx \kappa g^{d+1}/(2(d+1)T^d \rho^d)$, which is proportional to T at the relative accuracy $g = T\varepsilon$. The convex minorant $\hat{\psi}_t$ replaces φ_t because the conditional Jensen step in the proof of Theorem 3.5 needs Ψ_T convex.

3.2 The master theorem

In expectation, an iteration with a large gap raises the potential by a definite amount, and the potential is bounded, so there can be only few such iterations. The two versions of the theorem match the two conditions of Section 2.4; its high-probability statements rest on the following bound.

Lemma 3.4 (Decreasing bounded processes; Efroni et al., 2019, Thm. 3). *Let $(Y_k)_{k \geq 0}$ be adapted to $(\mathcal{A}_k)$ and nonincreasing, with $0 \leq Y_k \leq Y_0 = C$ and C deterministic, and let $\text{Reg}_K := \sum_{k \in [K]} (Y_{k-1} - \mathbb{E}[Y_k | \mathcal{A}_{k-1}])$. For $\delta \in (0, 1]$, with probability at least $1 - \delta$, $\text{Reg}_K \leq 9C \ln(3/\delta)$ for all K . In particular, if $\mathbb{E}[Y_{k-1} - Y_k | \mathcal{A}_{k-1}] \geq f_k \geq 0$ with f_k $\mathcal{A}_{k-1}$ -measurable, then, since $\text{Reg}_K \geq \sum_{k \in [K]} f_k$, $\sum_{k \geq 1} f_k \leq 9C \ln(3/\delta)$ with probability at least $1 - \delta$.*

In the theorem, Ψ_T is the aggregated lift function of Section 3.1, $\bar{\Gamma}_k$ and Γ are the error sums (9), and $\bar{U}_t$ and M_t are the Lipschitz envelope of the ceiling and the potential bound of (O3); if $\bar{U}_t$ is not Borel, its integral is an upper integral (Appendix A.4).

Theorem 3.5 (Master theorem). *Let Assumptions A and B hold.*

(a) Volume version. *If (VG_t) holds for all $t < T$, then*

$$\mathbb{E} \sum_{k \geq 1} \Psi_T((G^k - \bar{\Gamma}_k)_+) \leq \sum_{t < T} \int (\bar{U}_t - \underline{V}_t^0) d\mu_t \leq \sum_{t < T} M_t. \quad (14)$$

In particular, with deterministic errors, $\mathbb{E} \#\{k \geq 1 : G^k > \theta + \Gamma\} \leq \sum_{t < T} M_t / \Psi_T(\theta)$ for every $\theta > 0$. Moreover, for every $\delta \in (0, 1]$, with probability at least $1 - \delta$, simultaneously for all $\theta > 0$, $\#\{k \geq 1 : G^k > \theta + \bar{\Gamma}_k\} \leq 9 \ln(3/\delta) \sum_{t < T} M_t / \Psi_T(\theta)$.

- (b) Covering version. Assume deterministic errors γ_t, γ_t^F . Fix $\theta > 0$, numbers $\sigma_t \geq 0$ with $\sum_{t < T} L_t \sigma_t \leq \theta/4$, and finite σ_t -nets $C_t \subset X_t$. Then

$$\mathbb{E} \sum_{k \geq 1} (G^k - \Gamma - \theta/2)_+ \leq \sum_{t < T} \sum_{c \in C_t} (\bar{U}_t - \underline{V}_t^0)(c) \leq \sum_{t < T} |C_t| M_t. \quad (15)$$

In particular, $\mathbb{E} \#\{k \geq 1 : G^k > \theta + \Gamma\} \leq \frac{2}{\theta} \sum_{t < T} |C_t| M_t$. Moreover, for every $\delta \in (0, 1]$, with probability at least $1 - \delta$, $\#\{k \geq 1 : G^k > \theta + \Gamma\} \leq \frac{18 \ln(3/\delta)}{\theta} \sum_{t < T} |C_t| M_t$.

- (c) Finite state sets. Assume deterministic errors, every X_t finite and $\text{dist}_t = +\infty$ off the diagonal. Then $\mathbb{E} \sum_{k \geq 1} (G^k - \Gamma)_+ \leq C_{\text{fin}} := \sum_{t < T} \sum_{x \in X_t} (U_t - \underline{V}_t^0)(x)$ and, for every $\delta \in (0, 1]$, with probability at least $1 - \delta$, $\sum_{k \geq 1} (G^k - \Gamma)_+ \leq 9 \ln(3/\delta) C_{\text{fin}}$.

Part (a) handles all thresholds at once, allows random errors and has the better constants; Section 5.3 uses it, and Theorem 6.3 extends all three parts to the infinite horizon. Part (b) needs no volume growth, hence no convexity of the state sets; its proof (Appendix A.2) is the covering argument of Domingues et al. (2021, App. F), with (O2) in place of the Bellman update. It gives all the orders of this paper, and it is the version used for TFDP's generic cuts and Zhang and Sun's algorithm (Section 4). Part (c) is used for SDDiP. In part (c), every function on X_t is Borel, bounded and L_t -Lipschitz, so (O1) only asks $\underline{V}_t^0$ to be deterministic, and $\bar{U}_t = U_t$.

Proof of (a). Let $\Phi^k := \sum_{t < T} \int (\underline{V}_t^k - \underline{V}_t^0) d\mu_t$ and $C_U := \sum_{t < T} \int (\bar{U}_t - \underline{V}_t^0) d\mu_t$. By Lemma 3.1(i), $0 = \Phi^0 \leq \Phi^{k-1} \leq \Phi^k \leq C_U \leq \sum_{t < T} M_t$.

Pathwise lift. Fix $k \geq 1$ and $t < T$, let $x = \mathbf{x}_t^k$, and let $s_t := (r_t^k(x) - \gamma_t^k)_+$. As noted after Lemma 3.3, $\int (\underline{V}_t^k - \underline{V}_t^{k-1}) d\mu_t \geq \kappa_t \varphi_t(s_t) \geq \psi_t(s_t)$. Sum over $t < T$. By the definition of $\square$, by $\sum_{t < T} (a_t)_+ \geq (\sum_{t < T} a_t)_+$ and because Ψ_T is nondecreasing,

$$\Phi^k - \Phi^{k-1} \geq \sum_{t < T} \psi_t(s_t) \geq \Psi_T\left(\sum_{t < T} s_t\right) \geq \Psi_T((R^k - \Gamma_k)_+). \quad (16)$$

Conditional expectation. The map $(u, v) \mapsto \Psi_T((u - v)_+)$ is convex and finite on $\mathbb{R}^2$, R^k is bounded by Lemma 3.1(ii), and Γ_k is integrable. The conditional Jensen inequality, Lemma 3.2 and monotonicity give

$$\mathbb{E}[\Phi^k - \Phi^{k-1} \mid \mathcal{A}_{k-1}] \geq \Psi_T((\mathbb{E}[R^k \mid \mathcal{A}_{k-1}] - \mathbb{E}[\Gamma_k \mid \mathcal{A}_{k-1}])_+) \geq \Psi_T((G^k - \bar{\Gamma}_k)_+) =: f_k. \quad (17)$$

Summing over $k \in [K]$, the left-hand sides add up to $\mathbb{E}\Phi^K \leq C_U$; let $K \rightarrow \infty$ (monotone convergence).

High probability. Lemma 3.4 applies to $Y_k := C_U - \Phi^k$, which is adapted by (S3), with $Y_0 = C_U$ deterministic since $\underline{V}^0$, the envelopes $\bar{U}_t$ and the measures μ_t are, and to f_k , which is $\mathcal{A}_{k-1}$ -measurable. It gives $\sum_{k \geq 1} f_k \leq 9C_U \ln(3/\delta)$ with probability at least $1 - \delta$, and $\sum_k f_k \geq \Psi_T(\theta) \#\{k \geq 1 : G^k > \theta + \bar{\Gamma}_k\}$ for every θ . The last claim follows from (14), since Ψ_T is nondecreasing. $\square$

We now read the master theorem as statements about individual iterations.

Corollary 3.6 (Anytime guarantee). *In the setting of Theorem 3.5(a):*

- (a) If $\bar{\Gamma}_k \leq g_k$ for a deterministic sequence $(g_k)_{k \geq 1}$, then for all $\theta_1, \theta_2 > 0$, $\mathbb{E} \#\{k \geq 1 : G^k > \theta_1 + \theta_2\} \leq \#\{k \geq 1 : g_k > \theta_2\} + \sum_{t < T} M_t / \Psi_T(\theta_1)$.
- (b) With deterministic errors, let $K_\theta := 9 \ln(3/\delta) \sum_{t < T} M_t / \Psi_T(\theta)$. With probability at least $1 - \delta$, for every $\theta > 0$, $V_0(x_0) - \underline{V}_0^k(x_0) \leq \theta + \Gamma$ at every iteration $k \geq K_\theta$, and one of the policies $(\pi^k)_{1 \leq k \leq \lfloor K_\theta \rfloor + 1}$ is $(\theta + \Gamma + \zeta_0)$ -optimal, with ζ_0 as in Section 2.2.

- (c) (Almost sure convergence.) With deterministic errors, $\limsup_k G^k \leq \Gamma$ almost surely. In particular, with exact cuts, an exact forward pass and $U = V$, $G^k \rightarrow 0$ almost surely: the lower bound $\underline{V}_0^{k-1}(x_0)$ and the expected cost $V_0^{\pi^k}(x_0)$ of the policy both converge to $V_0(x_0)$.
- (d) (Uniformly selected iterate; no estimate of G^k is needed.) For $K \geq 1$ and $\theta > 0$, if $\mathbf{k}$ is uniform on $[K]$ and independent of the run, then $\mathbb{P}(G^{\mathbf{k}} > \theta + \bar{\Gamma}_{\mathbf{k}}) \leq \sum_{t < T} M_t / (K \Psi_T(\theta))$; on the complementary event, $\pi^{\mathbf{k}}$ is $(\theta + \bar{\Gamma}_{\mathbf{k}} + \zeta_0)$ -optimal and $V_0(x_0) - \underline{V}_0^{\mathbf{k}-1}(x_0) \leq \theta + \bar{\Gamma}_{\mathbf{k}}$.

Proof. (a) If $g_k \leq \theta_2$ and $G^k > \theta_1 + \theta_2$, the k -th term of (14) is at least $\Psi_T(\theta_1)$. (b) On the event of Theorem 3.5, at most K_θ indices k have $G^k > \theta + \Gamma$, so some $k \leq \lfloor K_\theta \rfloor + 1$ has $G^k \leq \theta + \Gamma$; use the gap inequality (8) and the monotonicity of $\underline{V}_0^j(x_0)$ in j . (c) For $\theta > 0$, $\Psi_T(\theta) > 0$, so $\#\{k \geq 1 : G^k > \theta + \Gamma\}$ is integrable by Theorem 3.5(a), hence finite almost surely; intersect these events over rational θ and use the gap inequality (8) with $\zeta_0 = 0$. (d) Given the run, the event $\{G^{\mathbf{k}} > \theta + \bar{\Gamma}_{\mathbf{k}}\}$ has conditional probability $\#\{k \in [K] : G^k > \theta + \bar{\Gamma}_k\} / K$, and each such k contributes at least $\Psi_T(\theta)$ to (14); use (8) on the complement. $\square$

We call $\bar{\Gamma}_k$ (or Γ , for deterministic errors) the *error floor*; it cannot be lowered in general (Remark A.3.1).

3.3 Explicit rates

We now make Ψ_T and M_t explicit, which gives the rates of Table 1. For a finite horizon the rates are stated for the relative accuracy $T\varepsilon$, an average error of ε per stage, the natural scale since the losses add up over T stages; replacing ε with ε/T gives the absolute accuracy ε (Remark 3.11).

The rates use one scale $\bar{\ell}$: the *initial error is at most $\bar{\ell}$ per stage* if

$$\sup_{X_t} (V_t - \underline{V}_t^0) \leq (T - t) \bar{\ell} \quad \text{for every } t < T. \quad (18)$$

This holds when every stage loss ℓ_t takes values in an interval $[a_t, a_t + \bar{\ell}]$ and $\underline{V}_t^0$ is the constant $\sum_{s>t} a_s$. With $U_t = V_t$, (18) gives $M_t \leq (T - t)\bar{\ell}$, because $\bar{U}_t \leq U_t$.

Corollary 3.7 (Stage-uniform constants). *In the setting of Theorem 3.5(a), assume $L_t \leq L$, $D_t \leq D$ and $\kappa_t \geq \kappa$, an initial error at most $\bar{\ell}$ per stage, $U_t = V_t$ and deterministic errors, and let $\varepsilon \leq 4LD$. Then $\Psi_T(g) \geq \kappa T \psi(g/T)$, where ψ is $\hat{\psi}_t$ with $\rho = 4LD$, and*

$$\mathbb{E} \#\{k \geq 1 : G^k > T\varepsilon + \Gamma\} \leq \frac{(d+1) \bar{\ell} (T+1) (4LD)^d}{\kappa \varepsilon^{d+1}}. \quad (19)$$

Proof. First, $\hat{\psi}_t \geq \psi$, because $\hat{\psi}_t$ is nonincreasing in ρ_t on both branches, and $\square_t(\kappa_t \hat{\psi}_t) \geq \kappa \square_t \hat{\psi}_t$. Jensen gives $\square_t \psi(g) \geq T \psi(g/T)$. By (18), $M_t \leq (T - t)\bar{\ell}$, so $\sum_{t < T} M_t \leq \bar{\ell} T(T+1)/2$. Finally $T \psi(\varepsilon) = T \varepsilon^{d+1} / (2(d+1)(4LD)^d)$; the condition $\varepsilon \leq 4LD$, the regime of interest, selects the first branch. $\square$

Remark 3.8 (Lipschitz constants that grow with the horizon). Assume exact valid cuts, $\kappa_t = 1$, an initial error at most $\bar{\ell}$ per stage, $D_t = D$ (no restriction when $D_t \leq D$, since increasing D_t only weakens (VG_t)) and $L_t = (T - t)\tilde{L}$, as relatively complete recourse typically gives (Proposition 4.1, with $\tilde{L} = \bar{\ell}/\eta$). Let $w_t := T - t$, $\bar{w} := \sum_{t < T} w_t = T(T+1)/2$, and let $\tilde{\psi}$ be built as in (13) with $\tilde{\rho} := 4\tilde{L}D$. Since $\rho_t = w_t \tilde{\rho}$, $\psi_t(s) = w_t \psi(s/w_t)$ is the perspective of $\tilde{\psi}$ with weight w_t , so $\Psi_T(g) = \bar{w} \tilde{\psi}(g/\bar{w})$, attained at $s_t = w_t g / \bar{w}$. For $\varepsilon \leq 4\tilde{L}D$ this gives

$$\mathbb{E} \#\{k \geq 1 : G^k > T\varepsilon\} \leq \frac{(d+1) \bar{\ell} (4\tilde{L}D)^d (T+1)^{d+1}}{2^d \varepsilon^{d+1}}, \quad (20)$$

and, with these L_t , the deterministic bound of [Zhang and Sun \(2022, Thm. 2\)](#) with per-stage tolerances ε is $\sum_{t < T} (1 + 2(T - t)\tilde{L}D/\varepsilon)^d$.

Corollary 3.9 (Discounted costs, horizon-free). *Let the stage- t loss be $\beta^{t-1}\hat{\ell}_t$ with $0 < \beta < 1$ and $\hat{\ell}_t$ taking values in an interval $[a_t, a_t + \bar{\ell}]$, let $\underline{V}_t^0$ be the constant $\sum_{s>t} \beta^{s-1}a_s$ (e.g. $\hat{\ell}_t \in [0, \bar{\ell}]$ and $\underline{V}^0 = 0$), let the cuts of index t be $\beta^t\hat{L}$ -Lipschitz, X_t convex ($\kappa_t = 1$), $D_t \leq D$, cuts valid and exact ($U = V$, $\gamma = \gamma^F = 0$), and $(1 - \beta)\varepsilon \leq 4\hat{L}D$. Then, for every horizon T ,*

$$\mathbb{E} \#\{k \geq 1 : G^k > \varepsilon\} \leq \frac{2(d+1)\bar{\ell}(4\hat{L}D)^d}{(1-\beta)^{d+2}\varepsilon^{d+1}}. \quad (21)$$

Proof. Increasing D_t only weakens $(\mathbf{VG}_t)$, so we may take $D_t = D$. Then $\rho_t = \beta^t\hat{\rho}$ with $\hat{\rho} := 4\hat{L}D$, so $\psi_t(s) = \beta^t\hat{\psi}(s/\beta^t)$, where $\hat{\psi}$ is built with $\hat{\rho}$. Jensen with weights β^t/B , $B = \sum_{t < T} \beta^t$, gives $\Psi_T(g) \geq B\hat{\psi}(g/B)$. The map $B \mapsto B\hat{\psi}(g/B)$ is nonincreasing, because $\hat{\psi}(s)/s$ is nondecreasing, and $B \leq (1 - \beta)^{-1}$; so $\Psi_T(g) \geq (1 - \beta)^{-1}\hat{\psi}((1 - \beta)g)$. Finally $M_t \leq \sup(V_t - \underline{V}_t^0) \leq \sum_{s>t} \beta^{s-1}\bar{\ell} \leq \beta^t\bar{\ell}/(1 - \beta)$, so $\sum_{t < T} M_t \leq \bar{\ell}/(1 - \beta)^2$. $\square$

With errors and general κ_t , the same proof gives (21) with $\varepsilon + \Gamma$ in the event and the bound divided by $\kappa = \min_t \kappa_t$.

Corollary 3.10 (Covering rates; finite state sets). *In the setting of [Theorem 3.5\(b\)](#) for (a) and of [Theorem 3.5\(c\)](#) for (b), with initial error at most $\bar{\ell}$ per stage and $U_t = V_t$:*

- (a) *if $L_t \leq L$ and X_t has σ -nets of size at most $Z(1 + 2D/\sigma)^d$ for every $\sigma > 0$ ([Lemma A.1.1](#); $Z \geq \max_t |X_t|$ bounds the number of discrete configurations, $Z = 1$ for convex X_t), then*

$$\mathbb{E} \#\{k \geq 1 : G^k > T\varepsilon + \Gamma\} \leq Z\bar{\ell}(T+1)(1 + 8LD/\varepsilon)^d/\varepsilon; \quad (22)$$

- (b) *if every X_t is finite, $\mathbb{E} \sum_k (G^k - \Gamma)_+ \leq \bar{\ell} \sum_{t < T} (T - t)|X_t|$, hence $\mathbb{E} \#\{k \geq 1 : G^k > \theta + \Gamma\} \leq \bar{\ell} T(T+1) \max_t |X_t|/(2\theta)$.*

Proof. (a) If $L > 0$, take $\sigma_t = \theta/(4TL)$; if $L = 0$, take a finite σ_t so large that $Z(1 + 2D/\sigma_t)^d < Z + 1$, so that X_t has a σ_t -net of size at most Z . In both cases $\sum_t L\sigma_t \leq \theta/4$ and, for $\theta = T\varepsilon$, $|C_t| \leq Z(1 + 8LD/\varepsilon)^d$; use $\sum_{t < T} M_t \leq \bar{\ell} T(T+1)/2$. (b) $M_t \leq (T - t)\bar{\ell}$ by (18), and $\bar{U}_t = U_t$. $\square$

For a finite state set X , horizon T and losses in $[0, 1]$, part (b) gives $\mathbb{E} \sum_k G^k \leq |X|T(T+1)/2$, the analogue of [Efroni et al. \(2019, Thm. 4\)](#) for RTDP. No Lipschitz property is used there, so every cut family that satisfies tightness (O2) and the ceiling condition (O3) qualifies ([Section 4.3](#)).

Remark 3.11 (Absolute accuracy). Replacing ε with ε/T in (19) gives, for $\varepsilon \leq 4LDT$,

$$\mathbb{E} \#\{k \geq 1 : G^k > \varepsilon + \Gamma\} \leq \frac{(d+1)\bar{\ell}T^{d+1}(T+1)(4LD)^d}{\kappa\varepsilon^{d+1}}, \quad (23)$$

where the extra factor T^{d+1} only reflects the per-stage error ε/T . With the constants of [Remark 3.8](#), the same substitution in (20) gives the bound $2(d+1)\bar{\ell}(4\tilde{L}D)^d\bar{w}^{d+1}/\varepsilon^{d+1}$, and that of [Zhang and Sun \(2022, Thm. 2\)](#), with per-stage tolerances ε/T , is $\sum_{t < T} (1 + 2(T - t)\tilde{L}DT/\varepsilon)^d$. [Corollary 3.10\(a\)](#) converts likewise.

3.4 Remarks and variants

Sharper and other forms of the bound. The middle term of (14) is an average and can be much smaller than $\sum_{t < T} M_t$; any probability measure satisfying $(\mathbf{VG}_t)$ can be used. Hölder's inequality also turns the master bound into a regret bound, $\mathbb{E} \sum_{k \in [K]} G^k = O(K^{d/(d+1)})$ ([Remark A.3.2](#)).

Multi-cut models. In a multi-cut model, $\underline{V}_t^k = \mathbb{E}_{\xi \sim \mathbb{P}_{t+1}} Q_{t,\xi}^k$, where $Q_{t,\xi}^k$ is the maximum of the cuts of scenario ξ . The monotone update (O4) then fails, but the proof of Theorem 3.5(a) only uses $\underline{V}_t^{k-1} \leq \underline{V}_t^k \leq \bar{U}_t$, Lipschitz approximations and the lift at the trial point. If each scenario cut at $x = \mathbf{x}_t^k$ is below the scenario value $\dot{\mathcal{B}}_t(\underline{V}_{t+1}^k; \cdot, \xi)$ of (4) and exact at x , the lift of the averaged model at x is at least $r_t^k(x)$ and $\underline{V}_t^k \leq V_t$, so the theorem applies with L_t the $\mathbb{P}_{t+1}$ -average of the per-scenario Lipschitz constants.

Several forward paths. Several forward paths per iteration lift the approximations at least as much as one path, so the bounds still hold, although they do not improve with the number of paths. Backward passes on the not-yet-updated approximation satisfy (O2) directly. With inexact forward passes, the guarantees concern the γ^F -greedy selections actually used; they do not transfer to a different, even exact, selection used at exploitation time (Forcier and Leclère, 2023, Sect. 2.1).

What is not covered. The analysis does not cover deterministic selection rules such as the problem child, for which the packing argument of Lan (2022a) and Zhang and Sun (2022) is the right tool, nor cut deletion, which breaks the monotone update (O4).

4 The main cut families

We now check Assumption B for the cut families of the TFDP framework; Table 2 gives the verdicts. Three are treated here. Benders cuts are those of SDDP. The generalised conjugacy cuts of Zhang and Sun give a partial answer to their question on bounds polynomial in T . SDDiP’s cuts on binary states use Theorem 3.5(c). The other families, and all proofs, are in Appendix B.

4.1 Benders cuts (SDDP)

Proposition 4.1 (Bounded subgradients). *Fix a norm and, for $\eta > 0$, let $X_t^\eta := \{x \in \text{aff } X_t : \text{dist}(x, X_t) \leq \eta\}$. Assume:*

- $\emptyset \neq \mathcal{X}_{t+1}(x, \xi) \subset X_{t+1}$ for $x \in X_t^\eta$ and $t < T$ (extended relatively complete recourse), and (S2) and (S3) hold with X_t^η in place of X_t as the set of incoming states;
- X_t is convex, the graph of $\mathcal{X}_{t+1}(\cdot, \xi)$ over X_t^η is convex and $\ell_{t+1}(\cdot, \cdot, \xi)$ is jointly convex on it;
- the approximations are convex, with $\underline{V}_t^0 \leq V_t$ and $\underline{V}_t^0 \leq \mathcal{B}_t(\underline{V}_{t+1}^0)$;
- every cut is a Benders cut $f_t^k(y) = h(x) + \langle g, y - x \rangle$ of $h := \mathcal{B}_t(\underline{V}_{t+1}^k)$ at $x = \mathbf{x}_t^k$, with g a subgradient of the restriction of h to $\text{aff } X_t$.

Then $V_t = \mathcal{B}_t(V_{t+1})$ is bounded on X_t^η , such g exist, and the cuts are valid on X_t^η , exact ($\gamma = 0$), and L_t -Lipschitz on X_t with

$$L_t \leq \left(\sup_{X_t^\eta} V_t - \inf_{X_t} \underline{V}_t^0 \right) / \eta. \quad (24)$$

If $\underline{V}_t^0 = \sum_{s>t} a_s$ and each ℓ_s takes values in $[a_s, a_s + \bar{\ell}]$ for incoming states in X_{s-1}^η (not only in X_{s-1}), this is at most $(T-t)\bar{\ell}/\eta$. So Assumption B holds with $U_t = V_t$, provided $\underline{V}_t^0$ is also L_t -Lipschitz (e.g. $\underline{V}_t^0$ constant).

This is the classical bounded-subgradient argument. The sets must be chosen so that $X_{t+1} \supset \mathcal{X}_{t+1}(X_t^\eta, \xi)$, so X_{t+1} may have to be larger than the set of reachable states, and D must bound the diameter of this larger set. Under the bounded-loss condition and with finitely supported noise, the results of Section 3 hold verbatim for SDDP, with the constant growing like $T-t$ (Remark 3.8). Adaptive partition-based cuts (Song and Luedtke, 2015; Siddig and Song, 2022; Forcier and Leclère,

family	(O1) constant	(O2) tight for	(O3) ceiling	verdict	where
Benders (SDDP)	osc/η	$\mathcal{B}_t$, exact	V_t	covered	Proposition 4.1
TFDP generic	L_t	$\mathcal{B}_t, \gamma_t$	V_t	covered from the second iteration	Proposition B.3.1
SLDP reverse norm	L_t	$\mathcal{B}_t$, exact	V_t if $L_t \geq \text{Lip } V_t$	without snapping	Proposition B.4.1
SLDP AL cuts	$\ \lambda\ _* + \varrho \leq L_t$	$\mathcal{B}_t$ if $L_t \geq \text{Lip } g$, else $\mathcal{B}_t^R$	V_t	covered if $L_t \geq \text{Lip } g$	Remark B.4.2
Zhang–Sun	$\sum_m p(l_\lambda + l_\varrho)$	$\mathcal{B}_t^R$, exact	V_t	covered	Theorem 4.3
SDDiP, binary strengthened	not needed	$\mathcal{B}_t$, exact	V_t	covered	Proposition 4.4
Benders Lagrangian, MI states	—	not tight	V_t	only in addition to a tight family	Section 4.3
MIDAS steps	fails	duality gap	V_t	not covered	Section 4.3
		$\mathcal{B}_t$, exact	$V_t + (T - t)\varepsilon$	without snapping	Theorem B.7.1

Table 2: Cut families and Assumption B. Forward passes are greedy (exact or γ^F -optimal) trajectories drawn from the law used in the operator of (O2). osc is the oscillation $\sup_{X_t^\eta} V_t - \inf_{X_t} \underline{V}_t^0$ of Proposition 4.1, $\mathcal{B}_t^R$ the regularised operator of Section 4.2, whose value functions V_t^R differ from V_t , λ and ϱ the multiplier and the penalty weight of the cut, $\sum_m p(l_\lambda + l_\varrho)$ Zhang and Sun’s Lipschitz bound (Proposition B.5.1), g the nodal value function of Remark B.4.2; for snapping and “from the second iteration”, see Section 4.4.

2022) are Benders cuts of an aggregation of the scenarios, refined until the aggregated cut coincides at the trial point with the Benders cut of the full problem; they are covered as well, and cuts from coarser partitions are valid and may be added.

Example 4.2 (Stage-wise absorption: slopes uniform in the horizon). In Proposition 4.1, assume that, for every ξ , $\mathcal{X}_{t+1}(\cdot, \xi)$ is constant on X_t^η and $\ell_{t+1}(\cdot, y, \xi)$ is c -Lipschitz there: a deviation of the incoming state is absorbed within the stage at unit cost at most c . Then $\mathcal{B}_t(W)$ is c -Lipschitz on X_t^η for every bounded W , so the Benders cuts are c -Lipschitz, uniformly in t , k and T . This holds, for instance, for a multi-item inventory with stocks in a box $X \subset \mathbb{R}^n$, bounded regular orders, and unbounded emergency orders and disposals at unit cost at most c (Appendix B.2). Corollary 3.7 with $L = c$, $d = n$ and $\kappa = 1$ then gives, for $\varepsilon \leq 4cD$, $\mathbb{E} \#\{k \geq 1 : G^k > T\varepsilon\} \leq (n+1)\bar{\ell}(T+1)(4cD)^n/\varepsilon^{n+1}$.

4.2 Generalised conjugacy cuts

Zhang and Sun (2022) extend SDDP to nonconvex, mixed-integer nonlinear problems on a recombining scenario tree, whose transition probabilities do not depend on the parent node, so that the noise is stagewise independent. The forward step of their Algorithm 3 solves a *regularised* nodal problem, in which the incoming state $\bar{x}$ may be replaced by any z at the price of a penalty $\varpi \mathcal{R}(\bar{x} - z)$; the backward step returns a generalised conjugacy cut, which is Lipschitz by construction. Under their exact-penalisation assumption, the regularised and original problems have the same optimal value. In our terms, the forward pass is greedy for the regularised Bellman operator $\mathcal{B}_t^R$, with value functions $V_t^R := \mathcal{B}_t^R(V_{t+1}^R)$, $V_T^R := 0$, so we apply Section 3 with $\mathcal{B}_t^R$ and V_t^R in the roles of $\mathcal{B}_t$ and V_t , and G^k is the regularised gap. The cuts are tight for $\mathcal{B}_t^R$, but they lie below the value functions V_t of the original problem: this is the ceiling (O3) with $U = V \neq V^R$. By exact penalisation, $V_0^R(x_0) = V_0(x_0)$,

so $\zeta_0 = 0$. [Proposition B.5.1](#) checks [Assumptions A](#) and [B](#), and [Appendix B.5](#) gives the dictionary between the two notations.

Let ι be the first index i such that the root solution of iteration $i + 1$ is ε -optimal in their sense, the quantity bounded in their [Theorem 3](#); our horizon T is theirs plus one ([Appendix B.5](#)).

Theorem 4.3 (A bound polynomial in T for Algorithm 3 of [Zhang and Sun](#)). *Consider Algorithm 3 under their Assumptions 1–3 and the parameter bounds of [Proposition B.5.1](#), with scenario paths sampled stage by stage from the transition probabilities, independently across stages and iterations (for uniform transition probabilities, their Assumption 4 implies this). Assume that, for $1 \leq s < T$, the approximations of index s are L -Lipschitz, X_s has dimension at most d and lies in a ball of diameter D , nodal costs lie in $[0, \bar{\ell}]$, and $V_s \leq (T - s)\bar{\ell}$ on X_s . Then*

$$\mathbb{E}[\iota] \leq \bar{\ell} T(T + 1) (1 + 8LD(T - 1)/\varepsilon)^d / \varepsilon. \quad (25)$$

In their [Remark 3](#), Zhang and Sun ask “whether there exists a complexity bound for Algorithm 3 that is polynomial in T in general”; [Theorem 4.3](#) gives one, in their setting and accuracy notion, when the paths are drawn from the transition law. [Proposition B.5.3](#) gives the general form, with stage-dependent constants and a high-probability version, and compares it with their deterministic bound. With uniform sampling on trees whose transitions are not uniform, our argument only gives a bound with a likelihood factor exponential in T , and the question remains open ([Remark B.5.4](#)).

4.3 SDDiP: Lagrangian and integer optimality cuts on binary states

Proposition 4.4 (SDDiP). *Consider SDDiP ([Zou et al., 2019](#)) with binary states $X_t \subset \{0, 1\}^{n_t}$, mixed-integer linear nodal problems with compact feasible sets, solved exactly, complete continuous recourse, losses in $[0, \bar{\ell}]$, $\underline{V}^0 = 0$, and forward scenarios sampled from the tree’s distribution. Use, at each index $t < T$, a Lagrangian cut with an optimal multiplier or an integer optimality cut; other valid cuts, such as strengthened Benders cuts, may be added. Then [Theorem 3.5\(c\)](#) applies with $U_t = V_t$ and $\Gamma = 0$:*

$$\mathbb{E} \sum_{k \geq 1} G^k \leq \bar{\ell} \sum_{t < T} (T - t) |X_t| \leq 2^n \bar{\ell} \frac{T(T + 1)}{2}, \quad n := \max_{t < T} n_t, \quad (26)$$

and, for $\delta \in (0, 1]$, with probability at least $1 - \delta$, $\sum_k G^k \leq 9 \ln(3/\delta) 2^{n-1} \bar{\ell} T(T + 1)$.

To our knowledge, these are the first random-sampling rates for SDDiP that are polynomial in T ([Zou et al. 2019](#) prove finite convergence without a rate); they are linear in the number $|X_t| \leq 2^n$ of states, hence exponential in the number n of binary state variables.

For *non-binary* mixed-integer states, Lagrangian cuts are not tight in general ([Zou et al., Example 1](#)): [\(O2\)](#) holds only with γ_t equal to the duality gap. Nonlinear cuts restore tightness there: ReLU Lagrangian cuts ([Deng and Xie, 2024](#); [Bansal and Küçükyavuz, 2026a](#)), used in a convergent multistage method by [Bansal and Küçükyavuz \(2026b\)](#), and Lagrangian cuts on an iteratively lifted binary state space ([Yang and Yang, 2025](#)). These works prove convergence without a rate; whether such cuts admit uniform Lipschitz constants, which would give [\(O1\)](#), is open.

4.4 Other families

TFDP’s generic cuts, Lipschitz, valid and γ_t -tight, satisfy [Assumption B](#) from the second iteration on: TFDP starts from $-\infty$, so the analysis takes the first cuts as initial approximations ([Proposition B.3.1](#)); the resulting rate is polynomial in T , unlike TFDP’s ([Table 3](#)). SLDP’s reverse-norm

cuts satisfy it whenever their constant L_t is at least the Lipschitz constant of the true V_t (Proposition B.4.1). Sampled SLDP and MIDAS (Philpott et al., 2020) replace a forward state by an earlier one within a distance $\omega > 0$ (“snapping”); the forward process is then not a trajectory of a policy, Lemma 3.2 does not apply, and these versions are not covered. Without snapping, SLDP is covered by Theorem 3.5, and MIDAS’s non-Lipschitz step cuts by a monotone variant of the volume argument (Theorem B.7.1).

5 Noise that is not finitely supported

Nothing in Section 3 uses finite support. The gap identity (Lemma 3.2) involves the forward law only; the backward pass enters only through tightness (O2) and the ceiling (O3). We discuss four ways to obtain such cuts when $\mathcal{B}_t$ is an integral over a continuous law: exact cuts under structure, valid inexact cuts, backward passes on a fixed SAA, and sampled cuts. In all of them the forward paths are drawn from the true law. Noise with unbounded support adds a tail term (Proposition C.5.1). Proofs are in Appendix C.

5.1 Exact cuts for the true law

If the cut added at x is an L_t -Lipschitz affine minorant of $\mathcal{B}_t(V_{t+1}^k)$ on X_t that is exact at x , for the true, possibly continuous law, then Lemma 2.2 gives Assumption B with $U = V$ and $\gamma = 0$, and Theorem 3.5 and its corollaries hold verbatim, with $V_0^{\pi^k}$, V and G computed under the true law. Expected Benders cuts are such cuts whenever the expectations of the scenario values and slopes can be computed exactly (Proposition C.1.1), for instance for linear stages and a piecewise-polynomial density (Remark C.1.2; see also Forcier and Leclère, 2022); the per-iteration cost, which grows with the number of cuts, is not bounded by our analysis. For laws without closed-form integrals, numerical quadrature falls under Section 5.2 if its error is one-sided and bounded, and under Section 5.3 otherwise.

5.2 Valid inexact cuts

Valid cuts that are γ_t^k -tight at the trial point, as in TFDP (Forcier and Leclère, 2023), satisfy Assumption B with $U = V$, so Theorem 3.5 applies with the error floor $\bar{\Gamma}_k$, which cannot be lowered in general (Remark A.3.1).

The most direct valid inexact cut is a lower (Jensen) discretisation (Proposition C.2.1): average exact affine scenario cuts at the conditional means of the cells of a partition $\mathcal{P}$ of Ξ_{t+1} . If the scenario value $\xi \mapsto \dot{\mathcal{B}}_t(W; y, \xi)$ of (4) is convex and L_t^ξ -Lipschitz for the convex models W (the class $\mathcal{W}_{t+1}^c$ of Appendix C.2, which contains every $\underline{V}_{t+1}^k$ when $\underline{V}^0$ is convex and the cuts are affine and valid), this cut is valid for $\mathcal{B}_t(W)$ and γ -tight with $\gamma \leq L_t^\xi e(\mathcal{P})$, where $e(\mathcal{P}) := \mathbb{E} \|\xi_{t+1} - \mathbb{E}[\xi_{t+1} \mid \mathcal{P}]\|$. Jensen-type validity also underlies backward resampling (Cruz et al., 2026), which changes the backward discretisation across iterations while keeping lower bounds for the continuous law; the bound on γ is what our analysis needs from such schemes.

Cuts from dual-feasible approximate solutions (Guigues, 2020) are also valid and inexact. Errors that decrease along a refinement schedule, with a deterministic bound $\bar{\Gamma}_k \leq g_k$, are handled by Corollary 3.6(a).

5.3 Forward passes from the true law, backward passes on a fixed SAA

This is common practice: the policy is trained on a discretisation and simulated under the original law (Lan and Shapiro, 2024, Rem. 5.3). Unlike the analyses of SDDP on the SAA problem itself (Lan, 2022a) and of the SAA size needed as a function of T (Shapiro, 2006; Park et al., 2022), the result below is an iteration bound for the original problem.

Let $\hat{\mathbb{P}}_t$ be a fixed law on Ξ_t , for instance the empirical law of N i.i.d. draws made before the run; everything below is conditional on $\hat{\mathbb{P}} = (\hat{\mathbb{P}}_t)_t$. Let $\hat{\mathcal{B}}_t := \mathcal{B}_t^{\hat{\mathbb{P}}_{t+1}}$, let $\hat{V}_t$ be the value functions of the problem with laws $\hat{\mathbb{P}}_t$, and assume that the cut added at x is an L_t -Lipschitz minorant of $\hat{\mathcal{B}}_t(V_{t+1}^k)$ on X_t that is exact at x , that $\underline{V}_t^0$ is L_t -Lipschitz with $\underline{V}_t^0 \leq \hat{\mathcal{B}}_t(\underline{V}_{t+1}^0)$, and that $\hat{M}_t := \sup_{X_t}(\hat{V}_t - \underline{V}_t^0) < \infty$. Two one-sided biases are needed: Δ_t^+ controls the lift and Δ_t^- controls the policy. Define

$$\mathcal{W}_{t+1} := \{W : X_{t+1} \rightarrow \mathbb{R} \text{ } L_{t+1}\text{-Lipschitz, } \underline{V}_{t+1}^0 \leq W \leq \hat{V}_{t+1}\} \quad (\mathcal{W}_T = \{0\}), \quad (27)$$

$$\Delta_t^+ := \sup_{x \in X_t, W \in \mathcal{W}_{t+1}} (\mathcal{B}_t(W) - \hat{\mathcal{B}}_t(W))(x),$$

$$\Delta_t^- := \sup_{x \in X_t} (\hat{\mathcal{B}}_t(V_{t+1}) - \mathcal{B}_t(V_{t+1}))(x). \quad (28)$$

Theorem 5.1 (SAA backward passes, true-law forward passes). *Let Assumption A hold for both laws, with Ξ_t carrying $\mathbb{P}_t$ and $\hat{\mathbb{P}}_t$, and assume (VG_t). Let $G^k = V_0^{\pi^k}(x_0) - \underline{V}_0^{k-1}(x_0)$, where $V_0^{\pi^k}$ is the expected cost under the true laws, and let Ψ_T be the aggregated lift function of Section 3.1 and $\mathbf{\Gamma}_k^F$ the forward-error sum (9). Then, conditionally on $\hat{\mathbb{P}}$,*

$$\mathbb{E} \left[\sum_{k \geq 1} \Psi_T \left(\left(G^k - \mathbf{\Gamma}_k^F - \sum_{t < T} (\Delta_t^+)_+ \right)_+ \right) \right] \leq \sum_{t < T} \hat{M}_t, \quad (29)$$

the high-probability statement of Theorem 3.5(a) holds with $\hat{M}_t$ in place of M_t , and, for every $k \geq 1$,

$$V_0(x_0) - \underline{V}_0^{k-1}(x_0) \leq G^k, \quad V_0^{\pi^k}(x_0) - V_0(x_0) \leq G^k + \sum_{t < T} (\Delta_t^-)_+. \quad (30)$$

Proof sketch. Validity for $\hat{\mathcal{B}}$ (Lemma 2.2(a) for the $\hat{\mathbb{P}}$ -problem) gives (O3) with ceiling $U = \hat{V}$, and $\underline{V}_{t+1}^{k-1} \in \mathcal{W}_{t+1}$. At $x = \mathbf{x}_t^k$, the cut satisfies $f_t^k(x) = \hat{\mathcal{B}}_t(\underline{V}_{t+1}^k)(x) \geq \hat{\mathcal{B}}_t(\underline{V}_{t+1}^{k-1})(x) \geq \mathcal{B}_t(\underline{V}_{t+1}^{k-1})(x) - \Delta_t^+$, which is (O2) with the deterministic error $(\Delta_t^+)_+ \geq 0$. Apply Theorem 3.5(a). The policy bound is in Appendix C.3. $\square$

The SAA thus enters only through an additive error per stage and the ceiling $U = \hat{V}$, with no likelihood ratio between the laws of whole paths.

Corollary 5.2 (Wasserstein and total-variation biases). *(a) Let Ξ_{t+1} be a subset of a normed space and W_1 the Wasserstein-1 distance for its norm. If, for all $x \in X_t$ and $W \in \mathcal{W}_{t+1} \cup \{V_{t+1}\}$, the map $\xi \mapsto \dot{\mathcal{B}}_t(W; x, \xi)$ is L_t^ξ -Lipschitz on Ξ_{t+1} , then $\Delta_t^\pm \leq L_t^\xi W_1(\mathbb{P}_{t+1}, \hat{\mathbb{P}}_{t+1})$. Hence, with $\Delta_W := \sum_{t < T} L_t^\xi W_1(\mathbb{P}_{t+1}, \hat{\mathbb{P}}_{t+1})$,*

$$\mathbb{E} \left[\sum_{k \geq 1} \Psi_T \left((G^k - \mathbf{\Gamma}_k^F - \Delta_W)_+ \right) \right] \leq \sum_{t < T} \hat{M}_t, \quad V_0^{\pi^k}(x_0) - V_0(x_0) \leq G^k + \Delta_W. \quad (31)$$

(b) If, for these x and W , $\dot{\mathcal{B}}_t(W; x, \cdot)$ takes values in an interval of length rg_t , then $\Delta_t^\pm \leq \text{rg}_t \text{TV}(\mathbb{P}_{t+1}, \hat{\mathbb{P}}_{t+1})$, where $\text{TV}(\mathbb{P}, \hat{\mathbb{P}}) := \sup_A |\mathbb{P}(A) - \hat{\mathbb{P}}(A)|$.

The Lipschitz condition in (a) holds, for instance, with additive noise in the dynamics and losses Lipschitz in the next state and the noise (Appendix C.3), and in the inventory model of Example 4.2 (Appendix B.2). For an i.i.d. SAA of size N , $W_1(\mathbb{P}_{t+1}, \widehat{\mathbb{P}}_{t+1})$ concentrates around its mean (Appendix C.3), which is at most of order $N^{-1/q}$ for a bounded noise of dimension $q \geq 3$ (Fournier and Guillin, 2015, Thm. 1): the bias Δ_W is linear in T under stage-uniform constants, but the rate $N^{-1/q}$ suffers from the curse of dimensionality in q . For state sets in affine spaces of dimension at most d , a covering argument over the state and the approximation replaces q by d : $\Delta_t^+ = O(N^{-1/(d+2)}\sqrt{\ln N})$ (Proposition C.3.1), and Δ_t^- , which involves only V_{t+1} , is $O(\sqrt{d \ln N/N})$ (Remark C.3.2), both with high probability.

Part (b) is useful when the two laws share their atoms, for instance for an SAA of a finitely supported law; it is uninformative for the empirical SAA of an atomless law, whose total variation distance to that law is one. Some bias is unavoidable, and with forward passes drawn from the SAA itself we have no guarantee for the true problem (Remark C.3.3).

5.4 Sampled cuts

Cuts built from a fresh sample at each backward step are valid only in expectation. Shifting each one down by the confidence radius $b_{k,t} := (\text{rg}_t + 2L_t D_t) \sqrt{\ln(4/\delta_{k,t})/N_{k,t}}$, where $N_{k,t}$ is the sample size, rg_t bounds the range in ξ of the scenario values and $\sum_{k,t} \delta_{k,t} = \delta'$, makes all cuts valid on an event A of probability at least $1 - \delta'$. On A , a stopped-potential argument gives the bounds of Theorem 3.5(a), in expectation and with high probability, with the error floor raised by $2 \sum_{t < T} b_{k,t}$, the potential bound $\sum_{t < T} \sup_{X_t} (V_t - \underline{V}_t^0)$ and constants changed by terms of order δ' (Theorem C.4.1). Sample sizes of order $g^{-2} \log k$ give a floor g , whatever the noise dimension.

No unconditional expectation bound follows: off A an invalid cut is never removed and can exceed V_t , so the ceiling (O3) can fail. Raw sampled cuts can fail it too: without the shift, SDDP can stall forever on a 0.1-suboptimal policy with $G^k = 0$ (Proposition C.4.3).

6 Infinite horizon

Corollary 3.9 is horizon-free, but its algorithm is non-stationary, with one approximation per stage. Infinite-horizon and periodic versions of SDDP keep a single cut model shared by all times (Warrington et al., 2019; Shapiro and Ding, 2020; Dowson, 2020; Nannicini et al., 2021), and Ju and Lan (2023) give the first complexity bounds for them. We analyse the shared-model algorithm for an abstract oracle and general noise, as Sections 2 and 3 do in finite horizon. Two things change: the gap identity runs over a geometric horizon (Lemma 6.1), and all the cuts of a pass lift the same model, so a pass is credited only with its largest residual (Lemma 6.2). For SDDP, the main instance, we also give Lipschitz constants uniform in the discount factor (Section 6.5). Proofs are in Appendix D.

6.1 Stationary problem, shared-model algorithm and oracle

A stationary problem has a state set $X \subset \mathbb{R}^n$, an initial state $x_0 \in X$, i.i.d. noises $(\xi_t)_{t \geq 1}$ with law $\mathbb{P}$ on a measurable space Ξ , a next-state map $\mathcal{X}$, a loss ℓ and a discount factor $\beta \in (0, 1)$. For bounded W on X , $x \in X$ and $\xi \in \Xi$, let

$$\dot{\mathcal{B}}(W; x, \xi) := \inf_{y \in \mathcal{X}(x, \xi)} [\ell(x, y, \xi) + \beta W(y)], \quad \mathcal{B}W(x) := \int \dot{\mathcal{B}}(W; x, \xi) \mathbb{P}(d\xi), \quad (32)$$

the operators of [Section 2.1](#) with the discount inside; $\mathcal{B}W$ is defined when $\xi \mapsto \dot{\mathcal{B}}(W; x, \xi)$ is measurable, e.g. on the class $\mathfrak{M}$ of [Appendix D.1](#).

Assumption C (Stationary problem). For all $x \in X$ and $\xi \in \Xi$:

- (S1 $_{\infty}$) X is compact and $\emptyset \neq \mathcal{X}(x, \xi) \subset X$ (forward invariance).
- (S2 $_{\infty}$) ℓ is Borel and $0 \leq \ell(x, y, \xi) \leq \bar{\ell}$ for $y \in \mathcal{X}(x, \xi)$.
- (S3 $_{\infty}$) (Measurability.) [Appendix A.4](#) holds for (32) and the models below, with parameters (possibly including γ_k^F) in a countable disjoint union of Euclidean spaces, as the number of cuts is random; $\mathcal{B}$ has a bounded Borel fixed point V with $\dot{\mathcal{B}}(V; \cdot, \cdot)$ jointly measurable.

A policy is a nonanticipative Borel rule $\mathbf{x}_{t+1} = \pi_t(\xi_{[t+1]}) \in \mathcal{X}(\mathbf{x}_t, \xi_{t+1})$, *stationary* if $\mathbf{x}_{t+1} = \pi(\mathbf{x}_t, \xi_{t+1})$; its cost is $V^\pi(x_0) := \mathbb{E} \sum_{t \geq 0} \beta^t \ell(\mathbf{x}_t, \mathbf{x}_{t+1}, \xi_{t+1})$. On $\mathfrak{M}$, $\mathcal{B}$ is monotone and a β -contraction, $0 \leq V \leq \bar{\ell}/(1 - \beta)$, and $V^\pi(x_0) \geq V(x_0)$ for every policy ([Lemma D.1.1](#)); [Appendix D.1](#) also gives a sufficient condition for (S3 $_{\infty}$), which is not needed for finitely supported noise and deterministic oracles.

Shared-model algorithm. The algorithm keeps one model $\underline{V}^{k-1} : X \rightarrow \mathbb{R}$. Let $\mathcal{A}_{k-1}$ be the σ -field generated by everything drawn during the first $k - 1$ iterations: noises, pass lengths and any internal randomness of the oracles, which must be independent of the noise and of the length of the current pass. The policy π^k is the stationary policy of a forward operator as in (5): it selects $y \in \mathcal{X}(x, \xi)$ with $\ell(x, y, \xi) + \beta \underline{V}^{k-1}(y) \leq \dot{\mathcal{B}}(\underline{V}^{k-1}; x, \xi) + \gamma_k^F$, for an $\mathcal{A}_{k-1}$ -measurable forward error $\gamma_k^F \geq 0$. Iteration $k \geq 1$:

- *Pass length.* Variant (D): $\tau_k := H$, fixed. Variant (G): τ_k geometric, $\mathbb{P}(\tau_k > s) = \beta^s$ for $s \geq 0$, independent of the past and of the noise of the pass (so $\mathbb{E}\tau_k = 1/(1 - \beta)$).
- *Forward pass.* $\mathbf{x}_0^k := x_0$ and $\mathbf{x}_{t+1}^k := \pi^k(\mathbf{x}_t^k, \xi_{t+1}^k)$ for $t < \tau_k - 1$, with fresh i.i.d. noise and π^k frozen.
- *Backward pass.* $W := \underline{V}^{k-1}$; for t from $\tau_k - 1$ down to 0, an oracle returns a cut $f : X \rightarrow \mathbb{R}$ at $\mathbf{x}_t^k$ for the *current* W , and $W := \max(W, f)$, possibly with further cuts ((O4 $_{\infty}$)). Then $\underline{V}^k := W$.

For comparison, variant (N) is finite-horizon SDDP on the H -stage truncation, with stage losses $\beta^{t-1}\ell$ and one model per stage ([Section 6.6](#)). The gap is $G^k := V^{\pi^k}(x_0) - \underline{V}^{k-1}(x_0)$, where $V^{\pi^k}(x_0)$ is the cost of π^k given $\mathcal{A}_{k-1}$, and the residual is $r^k := \mathcal{B}\underline{V}^{k-1} - \underline{V}^{k-1}$.

Assumption D (Lipschitz cut oracle, shared model). There are a metric dist on X , possibly extended, a constant $\hat{L} \geq 0$, a ceiling $U : X \rightarrow (-\infty, +\infty]$ and $\mathcal{A}_{k-1}$ -measurable errors $\gamma_{k,t} \geq 0$ ($k \geq 1, t \geq 0$) such that, for all $k \geq 1$:

- (O1 $_{\infty}$) $\underline{V}^0$ and every cut are Borel and $\hat{L}$ -Lipschitz for dist, and $\underline{V}^0$ is deterministic and bounded.
- (O2 $_{\infty}$) The cut f returned at $x = \mathbf{x}_t^k$, $t < \tau_k$, satisfies $f(x) \geq \mathcal{B}\underline{V}^{k-1}(x) - \gamma_{k,t}$.
- (O3 $_{\infty}$) U is deterministic, and $\underline{V}^0$ and every cut are below U on X ; the *potential bound* $M_U := \sup_X (\bar{U} - \underline{V}^0)$ is finite, where $\bar{U}(x) := \inf_{z \in X} \{U(z) + \hat{L} \text{dist}(x, z)\}$, with $\hat{L} \cdot \infty := \infty$.
- (O4 $_{\infty}$) Every update is $W := \max(\{W, f\} \cup \{f'_i\}_{i \in [m]})$ for some $m \geq 0$, where the additional cuts f'_i , if any, satisfy (O1 $_{\infty}$) and (O3 $_{\infty}$).

This is [Assumption B](#) for a single model; the errors are fixed before the pass because the proof needs them independent of τ_k . As in [Lemma 3.1\(i\)](#), the models are Borel, $\hat{L}$ -Lipschitz and bounded, and $\underline{V}^0 \leq \underline{V}^{k-1} \leq W \leq \underline{V}^k \leq \bar{U}$ for every intermediate model W of pass k . As in [Lemma 2.2](#), cuts below $\mathcal{B}W$ for the current W , with $\underline{V}^0 \leq \mathcal{B}\underline{V}^0$, give (O3 $_{\infty}$) with $U = V$ and $r^k \geq 0$, and cuts tight for the current W satisfy (O2 $_{\infty}$) ([Lemma D.1.2](#)).

What the gap measures. Let $\zeta := (U(x_0) - V(x_0))_+$, which is 0 if $U = V$. By [Lemma D.1.1\(iii\)](#) and [\(O3_∞\)](#), for every $k \geq 1$,

$$G^k \geq V(x_0) - \underline{V}^{k-1}(x_0) \quad \text{and} \quad G^k \geq V^{\pi^k}(x_0) - V(x_0) - \zeta. \quad (33)$$

So, when $\zeta = 0$, $G^k \leq \varepsilon$ means that the *stationary* policy π^k is ε -optimal from x_0 and the lower bound ε -tight; this implies Ju and Lan's guarantee, ε -optimality of the first decision ([Lemma D.1.1\(iii\)](#)).

Random horizon. If $\tau \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ is independent of the noise and $\mathbb{P}(\tau > t) = \beta^t$, the undiscounted loss paid before τ has expectation $V^\tau(x_0)$ ([Puterman, 1994](#)); so variant (G) is Dowson's passes on a one-node policy graph with a self-loop of probability β ([Dowson, 2020](#)).

6.2 Gap identity and lift of a pass

Let $\hat{\rho} := 4\hat{L}D$, with D and d the constants of the volume-growth condition ([VG_t](#)) on X , let $\hat{\varphi}$ and $\hat{\psi}$ be the functions φ_t and ψ_t of [Section 3.1](#) with $\rho_t = \hat{\rho}$ and $d_t = d$, and write $\mathbb{E}_{k-1} := \mathbb{E}[\cdot \mid \mathcal{A}_{k-1}]$.

Lemma 6.1 (Gap identity). *Under [Assumption C](#), let $\underline{V}^{k-1}$ be bounded, let $(\mathbf{x}_t)_{t \geq 0}$ be the trajectory of π^k from x_0 driven by i.i.d. noise independent of $\mathcal{A}_{k-1}$, $\lambda_t := \ell(\mathbf{x}_t, \mathbf{x}_{t+1}, \boldsymbol{\xi}_{t+1})$, and $e_t := \lambda_t + \beta \underline{V}^{k-1}(\mathbf{x}_{t+1}) - \dot{B}(\underline{V}^{k-1}; \mathbf{x}_t, \boldsymbol{\xi}_{t+1}) \in [0, \gamma_k^F]$ the forward error. For every $H \geq 0$,*

$$G^k = \mathbb{E}_{k-1} \left[\sum_{t < H} \beta^t (r^k(\mathbf{x}_t) + e_t) \right] + \varepsilon_H^k, \quad \varepsilon_H^k \leq \beta^H \left(\frac{\bar{\ell}}{1 - \beta} - \inf_X \underline{V}^{k-1} \right), \quad (34)$$

where $\varepsilon_H^k := \mathbb{E}_{k-1}[\sum_{t \geq H} \beta^t \lambda_t] - \beta^H \mathbb{E}_{k-1} \underline{V}^{k-1}(\mathbf{x}_H)$ is what a pass of length H misses. Hence $G^k = \mathbb{E}_{k-1} \sum_{t \geq 0} \beta^t (r^k(\mathbf{x}_t) + e_t)$, and, if $\tau \in \mathbb{N}^*$ is independent of the path given $\mathcal{A}_{k-1}$, with $\mathbb{P}(\tau > t \mid \mathcal{A}_{k-1}) = \beta^t$ for all $t \geq 0$, then exactly $G^k = \mathbb{E}_{k-1}[\sum_{t < \tau} (r^k(\mathbf{x}_t) + e_t)]$.

Lemma 6.2 (A pass on a shared model lifts by its largest residual). *Let [Assumptions C](#) and [D](#) hold and [\(VG_t\)](#) hold on X , for dist, with constants (μ, κ, D, d) . Let $\Phi^k := \int (\underline{V}^k - \underline{V}^0) d\mu$ and $s_t^k := (r^k(\mathbf{x}_t^k) - \gamma_{k,t})_+$. In variants (D) and (G), for every $k \geq 1$,*

$$\Phi^k - \Phi^{k-1} \geq \kappa \max_{t < \tau_k} \hat{\varphi}(s_t^k) \geq \kappa \hat{\psi}(\bar{s}^k), \quad \bar{s}^k := \max_{t < \tau_k} s_t^k, \quad (35)$$

and $0 = \Phi^0 \leq \Phi^{k-1} \leq \Phi^k \leq \bar{M}_U := \int (\bar{U} - \underline{V}^0) d\mu \leq M_U$.

Only the maximum is available because the regions lifted during a pass can coincide, e.g. if the path stays at one point. A summation by parts ([Lemma D.1.3](#)) turns this maximum back into the discounted sum of [Lemma 6.1](#): for τ independent of bounded $s_t \geq 0$, $\mathbb{E} \max_{t < \tau} s_t \geq (1 - \beta) \sum_t \beta^t s_t$ for all such s if and only if $\mathbb{P}(\tau > t) \geq \beta^t$ for all t , and among these laws the geometric one has the fewest expected cuts.

6.3 The master theorem in infinite horizon

Let $\Gamma_k^\beta := \sum_{t \geq 0} \beta^t \gamma_{k,t} + \gamma_k^F / (1 - \beta)$, the *error floor*, equal to $(\gamma + \gamma^F) / (1 - \beta)$ for constant errors. As in [Theorem 3.5](#), $\int (\bar{U} - \underline{V}^0) d\mu$ is an upper integral if $\bar{U}$ is not Borel.

Theorem 6.3 (Master theorem, infinite horizon). *Let [Assumptions C](#) and [D](#) hold, and run variant (G).*

(a) Volume version. If $(\mathbf{VG}_t)$ holds on X as in [Lemma 6.2](#), then

$$\mathbb{E} \sum_{k \geq 1} \kappa \hat{\psi}((1-\beta)(G^k - \Gamma_k^\beta)_+) \leq \bar{M}_U \leq M_U. \quad (36)$$

In particular, if $\Gamma_k^\beta \leq \Gamma^\beta$ for a deterministic Γ^β , then, for every $\theta > 0$, $\mathbb{E} \#\{k \geq 1 : G^k > \theta + \Gamma^\beta\} \leq \bar{M}_U / (\kappa \hat{\psi}((1-\beta)\theta))$, which equals $2(d+1)\bar{M}_U \hat{\rho}^d / (\kappa(1-\beta)^{d+1} \theta^{d+1})$ when $(1-\beta)\theta \leq \hat{\rho}$. Moreover, for every $\delta \in (0, 1]$, with probability at least $1 - \delta$, simultaneously for all $\theta > 0$, $\#\{k \geq 1 : G^k > \theta + \Gamma_k^\beta\} \leq 9 \ln(3/\delta) \bar{M}_U / (\kappa \hat{\psi}((1-\beta)\theta))$.

(b) Covering version. Fix $\theta > 0$, $\sigma \geq 0$ with $4\hat{L}\sigma \leq (1-\beta)\theta$, and a finite σ -net C of X for dist , and let $M_C := \sum_{c \in C} (\bar{U} - \underline{V}^0)(c) \leq |C| M_U$. Then

$$\mathbb{E} \sum_{k \geq 1} (G^k - \Gamma_k^\beta - \theta/2)_+ \leq \frac{M_C}{1-\beta}. \quad (37)$$

In particular, $\mathbb{E} \#\{k \geq 1 : G^k > \theta + \Gamma_k^\beta\} \leq 2M_C / ((1-\beta)\theta)$. Moreover, for every $\delta \in (0, 1]$, with probability at least $1 - \delta$, $\#\{k \geq 1 : G^k > \theta + \Gamma_k^\beta\} \leq 18 \ln(3/\delta) M_C / ((1-\beta)\theta)$.

(c) Finite state sets. If X is finite and $\text{dist} = +\infty$ off the diagonal, then $\mathbb{E} \sum_{k \geq 1} (G^k - \Gamma_k^\beta)_+ \leq C_{\text{fin}} / (1-\beta)$, with $C_{\text{fin}} := \sum_{x \in X} (U - \underline{V}^0)(x)$. Moreover, for every $\delta \in (0, 1]$, with probability at least $1 - \delta$, $\sum_{k \geq 1} (G^k - \Gamma_k^\beta)_+ \leq 9 \ln(3/\delta) C_{\text{fin}} / (1-\beta)$.

The proof ([Appendices D.2 and D.3](#)) follows that of [Theorem 3.5](#). We call (36) the *convex-regret form*; with $U = V$ and $\underline{V}^0 \geq 0$, $\bar{M}_U \leq \bar{\ell} / (1-\beta)$, as in [Corollary 3.9](#). In (b), if X lies in an affine space of dimension d , with dist given by a norm and $\text{diam } X \leq D$, the choice $\sigma = (1-\beta)\theta / (4\hat{L})$ gives $|C| \leq (1 + 8\hat{L}D / ((1-\beta)\theta))^d$ ([Lemma A.1.1](#)(a); one point if $\hat{L} = 0$). Since τ_k is independent of $\mathcal{A}_{k-1}$ and $\mathbb{E}\tau_k = 1/(1-\beta)$, the iterations counted in (a) and (b) contain in expectation $1/(1-\beta)$ times as many trial points, each with one cut of [\(O2_∞\)](#).

Remark 6.4 (On the factor $(1-\beta)^{-(d+1)}$). In [Theorem 6.3](#)(a), one factor $(1-\beta)^{-1}$ comes from the potential and $(1-\beta)^{-(d+1)}$ from evaluating $\hat{\psi}$ at $(1-\beta)\varepsilon$, that is, from the maximum over the pass. The factor $1-\beta$ inside $\hat{\psi}$ cannot be removed from a bound of this convex-regret form ([Proposition D.8.1](#), for $d = 1$): a value error of order r_0 can cause a policy error $r_0/(1-\beta)$, the classical loss of greedy policies ([Singh and Yee, 1994](#)). Whether the *count* of bad iterations can avoid it is open.

Deterministic pass length. Variant (D) with $H := \lceil \ln((d+2)\bar{\ell} / ((1-\beta)\theta)) / (1-\beta) \rceil$, $\underline{V}^0 \geq 0$, $\theta < \bar{\ell} / (1-\beta)$ and a deterministic floor Γ^β loses only a factor e : the tail of [Lemma 6.1](#) is at most $\theta / (d+2)$, and the count of part (a) holds with $2e(d+1)$ in place of $2(d+1)$, for one θ at a time ([Theorem D.4.1](#)). Geometric lengths remove the logarithm in H , the tail and the dependence of the pass length on θ .

6.4 Periodic problems

Seasonal problems, such as hydro-thermal scheduling with weekly stages and an annual cycle, are periodic ([Shapiro and Ding, 2020](#)). Let $S \geq 1$ be the period, $\mathbb{Z}_S := \mathbb{Z}/S\mathbb{Z}$ the set of *phases*, and $p(t) := t \bmod S$. Phase p has its own data X_p , $\mathbb{P}_p$ on Ξ_p , $\mathcal{X}_p(x, \xi) \subset X_{p+1}$ and ℓ_p , an operator $\mathcal{B}_p$ as in (32), and a model $\underline{V}_p^k$; the state $\mathbf{x}_t$ lies in $X_{p(t)}$, the noise ξ_{t+1} has law $\mathbb{P}_{p(t)}$, and $x_0 \in X_0$. At a time t of phase p , the policy π^k is γ_k^F -greedy for $\underline{V}_{p+1}^{k-1}$, and the cut at $\mathbf{x}_t^k$ is added to the model of phase p . The gap $G^k := V_0^{\pi^k}(x_0) - \underline{V}_0^{k-1}(x_0)$ is taken from phase 0, and the residuals are

$r_p^k := \mathcal{B}_p \underline{V}_{p+1}^{k-1} - \underline{V}_p^{k-1}$. We assume [Assumptions C](#) and [D](#) and $(\mathbf{VG}_t)$ phase by phase, with common $\hat{L}$, D and d , with $\mathcal{B}_{p(t)} \underline{V}_{p(t)+1}^{k-1}$ in [\(O2 \$_\infty\$ \)](#), and with, in [\(S3 \$_\infty\$ \)](#), a bounded Borel fixed point $V = (V_p)_p$ of the monotone β -contraction $(W_p)_p \mapsto (\mathcal{B}_p W_{p+1})_p$, each $\hat{\mathcal{B}}_p(V_{p+1}; \cdot, \cdot)$ being jointly measurable. The constants become $\kappa := \min_p \kappa_p$, $\Phi^k := \sum_p \int (\underline{V}_p^k - \underline{V}_p^0) d\mu_p$, $\bar{M}_U := \sum_p \int (\bar{U}_p - \underline{V}_p^0) d\mu_p$ and $\zeta := (U_0(x_0) - V_0(x_0))_+$; $S = 1$ is the stationary case. [Lemmas 6.1](#) and [6.2](#) hold with the same proofs, and [Lemma 6.2](#) improves, as each phase has its own model:

$$\begin{aligned} \Phi^k - \Phi^{k-1} &\geq \kappa \sum_{p \in \mathbb{Z}_S} \hat{\psi}(\bar{s}_p^k), \\ \bar{s}_p^k &:= \max\{(r_p^k(\mathbf{x}_t^k) - \gamma_{k,t})_+ : t < \tau_k, p(t) = p\} \quad (\max \emptyset := 0). \end{aligned} \quad (38)$$

Theorem 6.5 (Periodic problems, geometric pass length). *Under [Assumptions C](#) and [D](#) and $(\mathbf{VG}_t)$ phase by phase, the periodic algorithm with geometric pass lengths satisfies:*

- (i) $G^k = \mathbb{E}_{k-1}[\sum_{t < \tau_k} (r_{p(t)}^k(\mathbf{x}_t^k) + e_t)]$, with $e_t \in [0, \gamma_k^F]$, and $G^k \geq \max\{V_0^{\pi^k}(x_0) - V_0(x_0) - \zeta, V_0(x_0) - \underline{V}_0^{k-1}(x_0)\}$;
- (ii) $\mathbb{E} \sum_{k \geq 1} \kappa \chi_S((G^k - \Gamma_k^\beta)_+) \leq \bar{M}_U$, with $\chi_S(g) \geq \hat{\psi}((1 - \beta)g)$ defined in [\(53\)](#); with $q_S := S(1 - \beta)/(1 - \beta^S)$ and $\Gamma_k^\beta \leq \Gamma^\beta$ deterministic, for every $\theta > 0$ with $(1 - \beta)\theta \leq \hat{\rho}$,

$$\mathbb{E} \#\{k \geq 1 : G^k > \theta + \Gamma^\beta\} \leq \frac{2(d+1) \bar{M}_U \hat{\rho}^d}{\kappa(1 - \beta)^{d+1} \theta^{d+1}} \min\left\{1, \frac{q_S^{d+1}}{S}\right\}; \quad (39)$$

- (iii) for $\delta \in (0, 1]$, with probability at least $1 - \delta$, simultaneously for all $\theta > 0$, $\#\{k \geq 1 : G^k > \theta + \Gamma_k^\beta\} \leq 9 \ln(3/\delta) \bar{M}_U / (\kappa \chi_S(\theta))$.

[Theorem 6.3\(a\)](#) is the case $S = 1$, since $\chi_1(g) = \hat{\psi}((1 - \beta)g)$; parts (b) and (c) hold with one net per phase ([Appendix D.3](#)). Since $1 \leq q_S \leq \min\{S, 1 + S(1 - \beta)\}$, for SDDP ($\bar{M}_U \leq S\bar{\ell}/(1 - \beta)$, $\kappa = 1$, $\Gamma^\beta = 0$) the bound [\(39\)](#) is at most $2(d+1)\bar{\ell}(4\hat{L}D)^d \min\{S, (1 + S(1 - \beta))^{d+1}\} / ((1 - \beta)^{d+2} \theta^{d+1})$. This price of periodicity, the factor $\min\{S, q_S^{d+1}\}$ compared with $S = 1$, is at most S . It is close to 1 when $(d+1)S(1 - \beta)$ is small, since $q_S^{d+1} \leq e^{(d+1)S(1 - \beta)}$. For instance, weekly stages, an annual cycle and an annual discount rate of 5% give $S = 52$ and $q_S < 1.03$, so $q_S^{d+1} \approx 1.05$ for $d = 1$ but 1.3 for $d = 10$.

6.5 SDDP

For SDDP with exact Benders cuts, [Theorem 6.3](#) gives [Corollary 6.6](#), whose Lipschitz constant $\hat{L}$ we then bound, uniformly in β under a reachability condition ([Proposition 6.8](#)).

Assumption E (Shared-model SDDP). (I1) [Assumption C](#) holds, X is nonempty and convex, $d := \dim \text{aff } X$, $D \geq \text{diam } X$ (Euclidean norm), and avg_X is the average for Lebesgue measure on $\text{aff } X$ (the value at the point if $d = 0$). For each ξ , the graph $\{(x, y) : x \in X, y \in \mathcal{X}(x, \xi)\}$ is convex, $\mathcal{X}(x, \xi)$ is closed, and $\ell(\cdot, \cdot, \xi)$ is convex on the graph and lower semicontinuous in y . (I2) $\underline{V}^0 : X \rightarrow \mathbb{R}$ is convex and continuous, and $0 \leq \underline{V}^0 \leq \mathcal{B}\underline{V}^0$ (e.g. $\underline{V}^0 = 0$). (I3) The forward pass is exact ($\gamma_k^F = 0$), and a cut added at $x \in X$ to the current model W is an affine f with $f(x) = \mathcal{B}W(x)$ and $f \leq \mathcal{B}W$ on X (on X_η under [\(E \$_\eta\$ \)](#)). (I4) $\underline{V}^0$ and every cut are $\hat{L}$ -Lipschitz on X .

Condition (I1) makes the problem convex, and (I2)–(I3) describe SDDP with exact Benders cuts computed for the current model; the noise need not be finitely supported ([Section 5.1](#)). Condition (I4) assumes a Lipschitz constant for the cuts, which [Proposition 6.8](#) derives, as [Proposition 4.1](#) does in finite horizon, from extended relatively complete recourse:

(E_η) For some η > 0, with $X_η := \{z \in \text{aff } X : \text{dist}(z, X) \leq \eta\}$, the conditions of (I1) on the graph, $\mathcal{X}$ and ℓ hold with $X_η$ in place of X for the first argument, and (S1_∞)–(S3_∞) hold with $X_η$ in place of X as the set of incoming states; in particular $\emptyset \neq \mathcal{X}(z, \xi) \subset X$ and $\ell(z, y, \xi) \in [0, \bar{\ell}]$ for $z \in X_η$.

Condition (E_η) is used only there; it is stronger than Assumption 2.2 of Ju and Lan (2023), because successors of the neighbourhood must stay in X .

Corollary 6.6 (Shared-model SDDP, geometric pass length). *Under Assumption E, Assumption D holds with $U = V$, $\gamma_{k,t} = 0$, the Euclidean distance and $\hat{L}$ of (I4), and (VG_t) with $\kappa = 1$ (Lemma 2.3(a)); moreover $r^k \geq 0$, $\zeta = 0$ and $\bar{M}_U \leq \bar{M} := \text{avg}_X(V - \underline{V}^0) \leq \bar{\ell}/(1 - \beta)$. Hence variant (G) satisfies:*

- (i) $G^k = \mathbb{E}_{k-1}[\sum_{t < \tau_k} r^k(\mathbf{x}_t^k)]$ and $G^k \geq \max\{V^{\pi^k}(x_0) - V(x_0), V(x_0) - \underline{V}^{k-1}(x_0)\} \geq 0$;
- (ii) $\mathbb{E} \sum_{k \geq 1} \hat{\psi}((1 - \beta)G^k) \leq \bar{M}$, and, for every $\varepsilon > 0$ with $(1 - \beta)\varepsilon \leq \hat{\rho}$,

$$\mathbb{E} \#\{k \geq 1 : G^k > \varepsilon\} \leq \frac{2(d+1) \bar{M} \hat{\rho}^d}{(1 - \beta)^{d+1} \varepsilon^{d+1}} \leq \frac{2(d+1) \bar{\ell} (4\hat{L}D)^d}{(1 - \beta)^{d+2} \varepsilon^{d+1}}; \quad (40)$$

- (iii) for $\delta \in (0, 1]$, with probability at least $1 - \delta$, for all $\varepsilon > 0$, $\#\{k \geq 1 : G^k > \varepsilon\} \leq K_\varepsilon := 9 \ln(3/\delta) \bar{M} / \hat{\psi}((1 - \beta)\varepsilon)$ and $V(x_0) - \underline{V}^k(x_0) \leq \varepsilon$ for all $k \geq K_\varepsilon$.

Lipschitz constants. Under (E_η), the cuts of SDDP have a crude Lipschitz constant of order $\bar{\ell}/((1 - \beta)\eta)$, as in Proposition 4.1, which turns the bounds of Corollaries 3.9 and 6.6 into bounds of order $(1 - \beta)^{-(2d+2)} \varepsilon^{-(d+1)}$. A constant uniform in β needs a reachability condition.

Definition 6.7 (Robust reachability). Let $m \geq 1$ be an integer and $\chi \geq 0$. Condition $(R_{m,\chi})$ holds if, for every $z \in X_η$ and every $y \in X$, there are a policy started at z and a stopping time σ of the noise filtration such that $1 \leq \sigma \leq m$ and $\|\mathbf{x}_\sigma - y\| \leq \chi$ almost surely. Condition $(R_{1,0})$ means $\mathcal{X}(z, \xi) = X$ for all $z \in X_η$ and ξ .

Proposition 6.8 (Lipschitz bounds for the shared model). *Assume (I1)–(I3) and (E_η), and let $\underline{V}^0$ be L_0 -Lipschitz on X . In variants (D) and (G):*

- (i) every cut is $\bar{\ell}/((1 - \beta)\eta)$ -Lipschitz on X , so (I4) holds with $\hat{L} = \max\{L_0, \bar{\ell}/((1 - \beta)\eta)\}$;
- (ii) if $(R_{m,\chi})$ holds with $\chi < \eta$ and $L_0 \leq m\bar{\ell}/(\eta - \chi)$ (e.g. $\underline{V}^0 = 0$), every cut is $m\bar{\ell}/(\eta - \chi)$ -Lipschitz, so (I4) holds with $\hat{L} = m\bar{\ell}/(\eta - \chi)$, uniformly in β ;
- (iii) under $(R_{1,0})$, every cut is $\bar{\ell}/\eta$ -Lipschitz, so (I4) holds with $\hat{L} = \max\{L_0, \bar{\ell}/\eta\}$.

Under (E_η), (I3) asks for cuts valid on $X_η$, as subgradient cuts of the convex function $\mathcal{B}W$ on $X_η$ are (proof of Corollary 6.6). The β -uniform bound (ii) uses the shared model: the slope of a cut of $\mathcal{B}W$ is bounded by an oscillation of $\mathcal{B}W$ in which $\inf W$ cancels, which fails for stage-wise models (Appendix D.6; Remark D.6.2 gives examples of $(R_{m,0})$). For periodic problems, Proposition 6.8 holds phase by phase; its bound (ii) needs a reachability condition that returns to the right phase, which forces $m \geq S + 1$ unless one step suffices.

Corollary 6.9 (Relative accuracy, β -uniform constants). *Under (I1)–(I3) and (E_η) with $\underline{V}^0 = 0$ and $(R_{m,0})$, variant (G) satisfies, for $\varepsilon_{\text{rel}} \leq 4mD/\eta$,*

$$\mathbb{E} \#\left\{k : G^k > \varepsilon_{\text{rel}} \frac{\bar{\ell}}{1 - \beta}\right\} \leq \frac{2(d+1) (4mD/\eta)^d}{(1 - \beta) \varepsilon_{\text{rel}}^{d+1}}, \quad (41)$$

with $1/(1 - \beta)$ cuts per iteration in expectation. With the crude constant of Proposition 6.8(i), the right-hand side is $2(d+1)(4D/\eta)^d/((1 - \beta)^{d+1} \varepsilon_{\text{rel}}^{d+1})$, for $\varepsilon_{\text{rel}} \leq 4D/((1 - \beta)\eta)$.

Proof. [Corollary 6.6\(ii\)](#) with $\hat{L} = m\bar{\ell}/\eta$ ([Proposition 6.8\(ii\)](#)) and $\varepsilon = \varepsilon_{\text{rel}}\bar{\ell}/(1 - \beta)$; the condition $(1 - \beta)\varepsilon \leq \hat{\rho}$ reads $\varepsilon_{\text{rel}} \leq 4mD/\eta$. $\square$

This is the analogue of [Corollary 3.7](#), with T replaced by the effective horizon $1/(1 - \beta)$ and $\varepsilon_{\text{rel}}\bar{\ell}$ in the role of the per-stage error ε (ε_{rel} is a fraction of the largest cost $\bar{\ell}/(1 - \beta)$); for absolute accuracy, the factor $(1 - \beta)^{-(d+2)}$ of (40) matches the $T^{d+1}(T + 1)$ of (23).

6.6 Other oracles and non-stationary truncation

Other oracles. The oracles of [Sections 4](#) and [5](#) carry over to the shared model, with the constants and the part of [Theorem 6.3](#) given in [Proposition D.5.3](#). Valid inexact cuts and SAA backward passes with forward passes from the true law are covered ([Proposition D.5.1](#)), and so are shifted sampled cuts, through a stopped version of the proof ([Proposition D.5.2](#)); unbounded reachable sets are not. For SDDiP on binary states with exact forward passes, part (c) gives $\mathbb{E} \sum_k G^k \leq 2^n \bar{\ell}/(1 - \beta)^2$. The β -uniform constants of [Section 6.5](#) use convexity and affine cuts. For nonconvex oracles, a β -uniform constant is known only under stage-wise absorption ([Example 4.2](#)), where reverse-norm cuts with $\hat{L} = c$ qualify, and the regularisation constant of Zhang and Sun may have to grow with $1/(1 - \beta)$.

Non-stationary truncation. Variant (N) gives, by [Corollary 3.9](#) and a tail ε_H , the rate of variant (D) with H cuts per iteration, but it guarantees only a non-stationary policy that uses H functions ([Theorem D.7.1](#)). Using the normalised time-1 approximation $\mathcal{V}_1^k := \beta^{-1}V_1^k$ as a stationary model instead is not justified. On a deterministic instance, its greedy policy stays $\beta^2/(4(1 - \beta))$ -suboptimal at every iteration, while the truncated gap tends to zero for cut oracles with bounded slopes ([Proposition D.7.2](#)).

7 Discussion

7.1 Comparison with the SDDP literature

[Table 3](#) compares the bounds under stage-uniform constants. The accuracy notions differ; by [Remark 2.1](#), $G^k \leq \theta$ implies each of the others. Lan’s corrected bound uses his own recursion for the accuracy, with a factor $\prod_{t=2}^{T-1} N_t$ ([Lan, 2022b](#)). Iteration counts are comparable because an iteration costs $\Theta(\sum_t N_t)$ subproblems in all rows, for the deterministic Algorithm 2 of [Zhang and Sun \(2022, Rem. 2\)](#) as for SDDP (one forward subproblem and N_t backward subproblems per stage). The third row includes a factor 2^n absent from [Forcier and Leclère \(2023, Thm. 4.11\)](#): their proof drops a $+1$ in the bound on the number of effective iterations, as does that of their Cor. 4.7.

Comparing the upper bounds. Against the problem child, our bounds lose a factor of order $\bar{\ell}T/\theta$ at the accuracy θ , not a product over stages, whether the Lipschitz constants are uniform or grow like $T - t$ ([Table 3](#) and [Remarks 3.8](#) and [3.11](#)); at $\theta = T\varepsilon$ with uniform constants, this factor is about $(d + 1)2^d\bar{\ell}/\varepsilon$, independent of T . This ratio, which no known lower bound matches, comes from the potential: a region of X_t can receive up to M_t/θ lifts of size θ , whereas in the packing argument a saturated point stays saturated forever. For an absolute accuracy, linear dependence on T is not expected either: for $d \geq 5$, even the deterministic Algorithm 2 of Zhang and Sun needs a number of iterations superlinear in T , of order $(DL(T - 1)/(8\varepsilon))^{(d-2)/2}$, to certify ε -optimality with its stopping rule $\text{UB} - \text{LB} < \varepsilon$ ([Zhang and Sun, 2022, Thm. 5](#)). This is a lower bound for certified termination, not for the number of bad iterations of a randomised method, and it is consistent with our upper bound. Polynomiality in T also requires L_t and the state geometry to be controlled in T , which

forward pass	iterations, accuracy $T\varepsilon$	source
problem child / explorative	$T(1 + 2LD/\varepsilon)^d$	Zhang and Sun (2022, Cor. 2), Lan (2022a, Thm. 2), see also Forcier and Leclère (2023, Cor. 4.7)
uniform sampling, N_t atoms	$O(T(1 + 2LD/\varepsilon)^d \prod_{t < T} N_t)$	Zhang and Sun (2022, Thm. 3), Lan (2022b)
sampling, general law	$2^n(T - 1)(4DL(T - 1)/(T\varepsilon - \gamma_\Sigma))^{n+2(T-1)}$	Forcier and Leclère (2023, Thm. 4.11)
sampling (this paper)	$(d + 1) \bar{\ell}(T + 1)(4LD)^d/\varepsilon^{d+1}$	Corollary 3.7

Table 3: Finite horizon, stage-uniform constants, each state set in a ball of diameter D . Accuracy $T\varepsilon$ in the sense of each source, which $G^k \leq T\varepsilon$ implies (Remark 2.1); for an absolute accuracy ε , replace ε by ε/T in every row (Remark 3.11). Our bound counts all bad iterations, the others bound the first good one; only the first row certifies its output. γ_Σ is the total error and n the state dimension of Forcier and Leclère (2023).

can fail: the Hoffman recursion of SLDP can make L_t grow geometrically, and the regularisation weight of Zhang and Sun may have to grow with T (Zhang and Sun, 2022, after Assumption 2). Our bounds are exponential in d , unlike the dimension-free bounds of Barros and Guigues (2026); Barros et al. (2026) under strong convexity.

Infinite horizon. The guarantee of Ju and Lan (2023) is implied by $G^k \leq \varepsilon$ (Section 6.1). Counted in cuts, with a common β -uniform Lipschitz constant, our bound exceeds that of their explorative Inf-EDDP by a factor of order $1/\varepsilon_{\text{rel}}$, as in finite horizon, and that of their continual-exploration variant by a further $1/(1 - \beta)$, while against their sampled variant it improves from exponential in $1/(1 - \beta)$ to polynomial.

7.2 Comparison with reinforcement learning

SDDP can be seen as a convex, continuous-state version of RTDP (Barto et al., 1995). For finite MDPs with a known model, Efroni et al. (2019) prove the counterparts of our gap identity, high-probability lemma and finite-state bound, and a uniform-PAC corollary (their Corollary 5); their sign convention, for rewards and optimistic upper values, mirrors SDDP’s lower values. In the discounted case, for the variant of RTDP that follows the frozen greedy policy during an episode of geometric length and then backs up the visited states in reverse order (variant (G) with counting measure), Theorem 6.3(c) gives $\mathbb{E} \sum_k G^k \leq |X| \bar{\ell}/(1 - \beta)^2$; classical RTDP, which updates the value before acting, needs its own timing argument. Strehl et al. (2006) proved PAC bounds, in a different protocol, for a modified RTDP with a known model that updates only when the decrease exceeds $\varepsilon(1 - \beta)$, the factor $1 - \beta$ of Corollary 6.6, and asked whether the original RTDP is PAC, a question that Efroni et al. answered.

Compared with the covering argument of Domingues et al. (2021, App. F) (see also Sinclair et al., 2020, 2023), which Theorem 3.5(b) follows, our constants are larger by a factor of about $2^d/(d + 1)$; the volume version adds a convex bound valid for all thresholds at once, the average-gap bound of Remark A.3.2 and a potential measured by an average, and requires (VG_t) . Domingues et al. and Sinclair et al. also treat unknown models and state no known-model planning bound. The upper bounds of Domingues et al., minima of Lipschitz “cones” centred at visited states, are mirrored reverse-norm cuts, so the case of SLDP reverse-norm cuts with finitely supported noise can

essentially be read off their Appendix F, specialised to a known model. Finally, heuristic search value iteration (Smith and Simmons, 2004; Delage et al., 2024) also keeps convex piecewise-linear bounds, but explores deterministically and terminates through coverings whose size is exponential in the depth.

7.3 Conclusion

For TFDP-type algorithms with randomly sampled forward passes, the number of iterations whose gap exceeds an error of ε per stage is polynomial in the horizon, and linear in it under stage-uniform Lipschitz constants. The argument needs only the forward law in the gap identity, and Lipschitz cuts that are tight at the trial point and lie below a fixed ceiling. Three questions remain: whether the nonlinear cuts that restore tightness on mixed-integer states have uniformly bounded Lipschitz constants (Section 4.3), whether nonconvex oracles have constants uniform in β (Section 6.6), and whether the number of bad iterations can avoid the factor $1 - \beta$ of Remark 6.4.

Use of large language models. Large language models were used during this work to suggest proof ideas, to check arguments, and to help with writing. All statements and proofs have been verified manually by the author, who takes full responsibility for any remaining errors.

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A Proofs for Sections 2 and 3

A.1 Proofs of Lemmas 2.2, 2.3 and A.1.1

Proof of Lemma 2.2. (a) Induction on k and, within an iteration, backward induction on t . At $t = T$, $\underline{V}_T = V_T = 0$. If $\underline{V}_{t+1}^k \leq V_{t+1}$ on X_{t+1} , then by forward invariance (S1) and monotonicity, $f \leq \mathcal{B}_t(\underline{V}_{t+1}^k) \leq \mathcal{B}_t(V_{t+1}) = V_t$ on X_t , hence $\underline{V}_t^k \leq V_t$. By (S2) and $V_T = 0$, each V_t is bounded on X_t , so $M_t \leq \sup V_t - \inf \underline{V}_t^0 < \infty$. For the sub-Bellman inequality, a cut created at iteration $j \leq k$ satisfies $f \leq \mathcal{B}_t(\underline{V}_{t+1}^j) \leq \mathcal{B}_t(\underline{V}_{t+1}^k)$, because $\underline{V}_{t+1}^j \leq \underline{V}_{t+1}^k$ and $\mathcal{B}_t$ is monotone; with $\underline{V}_t^0 \leq \mathcal{B}_t(\underline{V}_{t+1}^0) \leq \mathcal{B}_t(\underline{V}_{t+1}^k)$ this gives $\underline{V}_t^k \leq \mathcal{B}_t(\underline{V}_{t+1}^k)$. (b) The operator $\mathcal{B}_t$ is monotone. $\square$

Proof of Lemma 2.3. (a) For $\lambda \in [0, 1]$, $x + \lambda(X_t - x) \subset X_t \cap \{y : \|y - x\| \leq \lambda D_t\}$ by convexity, and its relative volume is λ^{d_t} . Take $\lambda = \min\{1, s/D_t\}$. (b) If $x \in K^j$, the dist_t -ball of radius s contains $K^j \cap \{\|y - x\| \leq s\}$, whose μ_t -measure is at least $|I_t|^{-1} \min\{1, (s/\text{diam } K^j)^{\dim K^j}\} \geq |I_t|^{-1} \min\{1, (s/D_t)^{d_t}\}$ by (a). (c) Apply (b) to singletons. $\square$

Lemma A.1.1 (Net sizes). *Let $\sigma > 0$. (a) If X_t is a compact subset of an affine space of dimension d_t , with dist_t given by a norm and diameter at most D_t , then X_t has a σ -net of size at most $(1 + 2D_t/\sigma)^{d_t}$. (b) If $X_t = \bigcup_{z \in Z_t} \{z\} \times K_z$, with each K_z compact in an affine space of dimension at most d_t and of diameter at most D_t , and dist_t is the extended metric of Lemma 2.3(b), then X_t has a σ -net of size at most $|Z_t|(1 + 2D_t/\sigma)^{d_t}$.*

Proof of Lemma A.1.1. (a) A maximal σ -separated subset of X_t is a σ -net. The balls of radius $\sigma/2$ around its points, taken in the affine hull, are disjoint and lie in a ball of radius $D_t + \sigma/2$ centred at any point of X_t ; comparing volumes gives $((D_t + \sigma/2)/(\sigma/2))^{d_t}$. (b) Apply (a) to each fibre. $\square$

A.2 Proof of Theorem 3.5(b) and (c)

With θ , σ_t and C_t as in Theorem 3.5(b), so that $2 \sum_{t < T} L_t \sigma_t \leq \theta/2$, let $Y_k := \sum_{t < T} \sum_{c \in C_t} (\bar{U}_t - \underline{V}_t^k)(c)$. By Lemma 3.1(i), (Y_k) is adapted, nonincreasing and nonnegative, and Y_0 is deterministic. Fix $k \geq 1$ and $t < T$, let $x = \mathbf{x}_t^k$, and choose $c \in C_t$ with $\text{dist}_t(x, c) \leq \sigma_t$. By (O2) and (O4), $(\underline{V}_t^k - \underline{V}_t^{k-1})(x) \geq (r_t^k(x) - \gamma_t)_+$. Both approximations are L_t -Lipschitz, so $(\underline{V}_t^k - \underline{V}_t^{k-1})(c) \geq (r_t^k(x) - \gamma_t - 2L_t \sigma_t)_+$, and the lifts at the other net points are nonnegative. Hence, with R^k the residual sum (7),

$$Y_{k-1} - Y_k \geq \sum_{t < T} (r_t^k(\mathbf{x}_t^k) - \gamma_t - 2L_t \sigma_t)_+ \geq \left(R^k - \sum_{t < T} \gamma_t - \theta/2 \right)_+. \quad (42)$$

By Jensen for $(\cdot)_+$ and [Lemma 3.2](#), $\mathbb{E}[Y_{k-1} - Y_k \mid \mathcal{A}_{k-1}] \geq (G^k - \Gamma - \theta/2)_+$. Summing over $k \in [K]$ gives the first claim, since $Y_0 - \mathbb{E}Y_K \leq Y_0$. The count follows from $(\theta/2) \#\{k \geq 1 : G^k > \theta + \Gamma\} \leq \sum_{k \geq 1} (G^k - \Gamma - \theta/2)_+$. The high-probability claim is [Lemma 3.4](#) applied to (Y_k) , with $C = Y_0$ (deterministic because $\underline{V}_t$, $\bar{U}_t$ and the nets C_t are) and $f_k = (G^k - \Gamma - \theta/2)_+$. For (c), take $\sigma_t = 0$ and $c = x$, so that the Lipschitz property is not used, and repeat the computation with $\theta = 0$, $f_k = (G^k - \Gamma)_+$ and $C = C_{\text{fin}}$. $\square$

A.3 Error floor and regret

Remark A.3.1 (The error floor is attained). If every $X_t = \{\bar{x}_t\}$ is a point, $\ell_t \equiv \bar{\ell}$, $\underline{V}^0 \equiv 0$, and the oracle returns the valid, γ_t -tight constant cut $\mathcal{B}_t(\underline{V}_{t+1}^k)(\bar{x}_t) - \gamma_t$, with $0 \leq \gamma_t \leq \bar{\ell}$, then $G^k = \sum_{t < T} \gamma_t$ for all $k \geq 2$. If $T = 1$, $\mathcal{X}_1(x_0, \xi) = [0, 1]$, $\ell_1(x_0, y, \xi) = \gamma^F y$, $\underline{V}^0 \equiv 0$ and the forward oracle always returns $y = 1$, then $G^k = \gamma^F$ for all $k \geq 1$.

Remark A.3.2 (Regret). In the setting of [Corollary 3.7](#) with $\gamma = \gamma^F = 0$ and $\kappa = 1$, [Lemma 2.2\(a\)](#) and [Lemma 3.2](#) give $0 \leq G^k \leq M^V := \sum_{t < T} \sup_{X_t} (V_t - \underline{V}_t^0)$. The function ψ of [Corollary 3.7](#), with $\rho = 4LD$, satisfies $\psi(s) \geq s^{d+1}/(2(d+1)s_{\max}^d)$ on $[0, s_{\max}]$ when $s_{\max} \geq \rho$. Hence $\Psi_T(g) \geq g^{d+1}/(2(d+1)T^d \max\{\rho, M^V/T\}^d)$ for $g \leq M^V$, and Hölder's inequality with the master bound (14) gives

$$\mathbb{E} \sum_{k \in [K]} G^k \leq K^{d/(d+1)} (2(d+1)T^d \max\{\rho, M^V/T\}^d M^V)^{1/(d+1)}. \quad (43)$$

A.4 Conditions (S2) and (S3)

Condition (S2) is equivalent to the boundedness of $\mathcal{B}_t(0)$ when the losses are bounded below. For signed losses and noise that is not finitely supported, the boundedness of $\mathcal{B}_t(0)$ alone does not suffice: the cost of a greedy path can then fail to be integrable.

The Lipschitz envelope $\bar{U}_t$ of (O3) is Borel for a norm, for the extended metric of [Lemma 2.3\(b\)](#) and on finite sets. In general, in [Theorem 3.5\(a\)](#), $\int (\bar{U}_t - \underline{V}_t^0) d\mu_t$ denotes the upper integral $\inf\{\int g d\mu_t : g \text{ Borel}, g \geq \bar{U}_t - \underline{V}_t^0\}$, which suffices since the approximations are Borel by (O1) and lie below $\bar{U}_t$.

The approximations are of the form $\underline{V}_t^k = \mathbf{v}_t^k(\mathbf{p}_t^k, \cdot)$, with $\mathbf{p}_t^k$ a random vector of a Euclidean space $\mathbf{P}_k$ whose dimension may grow with k and $\mathbf{v}_t^k : \mathbf{P}_k \times X_t \rightarrow \mathbb{R}$ Borel, with $\mathbf{v}_t^k(\mathbf{p}, \cdot)$ bounded for every $\mathbf{p}$. For these functions, the map $(\mathbf{p}, x, \xi) \mapsto \dot{\mathcal{B}}_t(\mathbf{v}_{t+1}^k(\mathbf{p}, \cdot); x, \xi)$ is jointly measurable and, for every $\gamma > 0$, admits a jointly measurable γ -minimising selection on the set where it is finite, and for $\gamma = 0$ one when the infimum is attained. The map $(x, \xi) \mapsto \dot{\mathcal{B}}_t(W; x, \xi)$ is also jointly measurable for $W = V_{t+1}$ and for the bounded continuous test functions W of [Section 5](#). Joint measurability cannot be asked for every bounded Borel $\mathbf{v}_{t+1}^k$, since the projection of a Borel set need not be Borel. *Sufficient condition.* It holds when $(x, \xi; y) \mapsto \ell_{t+1}(x, y, \xi) + \delta_{\mathcal{X}_{t+1}(x, \xi)}(y)$ is a normal integrand and each $\mathbf{v}_{t+1}^k$ is Borel in $\mathbf{p}$ and continuous in $y \in \mathbb{R}^{n_{t+1}}$, as is a maximum of finitely many cuts with Borel coefficients that are Lipschitz for a norm. The sum is then a normal integrand with parameter $(\mathbf{p}, x, \xi)$, so its optimal value is measurable and its γ -argmin mappings ($\gamma \geq 0$) are measurable and closed-valued. They admit measurable selections where they are nonempty, which for $\gamma > 0$ is where the optimal value is finite ([Rockafellar and Wets, 1998](#), Ex. 14.29, Prop. 14.44(c), Thm. 14.37, Prop. 14.33 and Cor. 14.6). The same holds for every bounded lower semicontinuous W ([Rockafellar and Wets, 1998](#), Ex. 14.30), in particular for these test functions and for a Lipschitz V_{t+1} . For other cut families, such as the step cuts of [Appendix B.7](#), (S3) is an assumption.

B Cut families: complements and proofs for Section 4

B.1 Benders cuts

Proof of Proposition 4.1. By the standard argument (convex combinations of ϵ -optimal points stay feasible by graph convexity), $h = \mathcal{B}_t(\underline{V}_{t+1}^k)$ is convex on the convex set X_t^η . By induction, every cut of index $t+1$ is valid on X_{t+1} , so $\underline{V}_{t+1}^0 \leq \underline{V}_{t+1}^k \leq V_{t+1}$ there (Lemma 2.2(a)). Since $\mathcal{X}_{t+1}(z, \xi) \subset X_{t+1}$ for $z \in X_t^\eta$, monotonicity and (S2) on X_t^η give, on X_t^η , $-\infty < \mathcal{B}_t(0) + \inf \underline{V}_{t+1}^0 \leq h \leq \mathcal{B}_t(V_{t+1}) = V_t \leq \sup_{X_t^\eta} \mathcal{B}_t(0) + \sup_{X_{t+1}} V_{t+1} < \infty$. Moreover $h \geq \underline{V}_t^k \geq \underline{V}_t^0$ on X_t by the sub-Bellman inequality. For $x \in X_t$, X_t^η contains a relative neighbourhood of x in $\text{aff } X_t$, so the relative subdifferential is nonempty. The subgradient inequality gives validity on X_t^η , and tightness holds by construction. For a unit vector u in the direction space of $\text{aff } X_t$, $h(x + \eta u) \geq h(x) + \eta \langle g, u \rangle$, so $\langle g, u \rangle \leq (\sup_{X_t^\eta} V_t - \inf_{X_t} \underline{V}_t^0) / \eta$. Since $\pm u$ are both admissible, f_t^k is L_t -Lipschitz on X_t . Under the bounded-loss condition, the same upper bound on V_t and backward induction give $\sup_{X_t^\eta} V_t \leq \sum_{s>t} (a_s + \bar{\ell})$. $\square$

B.2 Details of Example 4.2

Absorption. Comparing the infima defining $\dot{\mathcal{B}}_t(W; z, \xi)$ and $\dot{\mathcal{B}}_t(W; z', \xi)$, over the same set, shows that $\mathcal{B}_t(W)$ is c -Lipschitz on X_t^η for every bounded W ; so the Benders cuts are c -Lipschitz ($\eta \langle g, v \rangle \leq h(x + \eta v) - h(x) \leq c\eta$ for unit directions v of $\text{aff } X_t$).

Inventory. With the norm $\|\cdot\|_1$, let $X_t = X := \prod_{i \in [n]} [-b'_i, b_i]$, $t \in [T]$, be the stocks of n items (negative stocks are backlogs), $x_0 \in X$, $\xi_t \in \Xi := \prod_i [0, \bar{\xi}_i]$ the demand, and $y = x + u + e - w - \xi$, with regular orders $u \in \prod_i [0, \bar{u}_i]$ and *unbounded* emergency orders $e \geq 0$ and disposals $w \geq 0$, at unit costs $c_{o,i}, c_{e,i}, c_{w,i} \geq 0$. Then $\mathcal{X}_{t+1}(x, \xi) = X$ for every $x \in \mathbb{R}^n$, and $\ell_{t+1}(x, y, \xi) = \text{oc}(y - x + \xi) + \text{hc}(y)$. Here $\text{hc} \geq 0$ is a continuous convex holding and backlog cost, and $\text{oc}(r) \geq 0$, the cheapest cost of a net order $u + e - w = r$, is convex (a linear programme in r) and c -Lipschitz, $c := \max_i \max\{c_{e,i}, c_{w,i}\}$ (absorb a change of r_i with e_i or w_i). With $\underline{V}^0 = 0$, the losses lie in $[0, \bar{\ell}]$, $\bar{\ell} := c(D + \|\bar{\xi}\|_1) + \max_X \text{hc}$, where $D := \sum_i (b_i + b'_i) = \text{diam } X$.

Continuous demand (Section 5). As $\dot{\mathcal{B}}_t(W; x, \xi) =: \vartheta_W(x - \xi)$ depends on $x - \xi$ only, it is c -Lipschitz in ξ for every bounded W and convex in ξ for convex W . So Corollary 5.2 applies with $\text{rg}_t = c\|\bar{\xi}\|_1$, and the conditions of Proposition C.2.1 and Assumption F hold on $\mathcal{W}_{t+1}^c$ with $L_t = L_t^\xi = c$, or $c\sqrt{n}$ for the Euclidean norms of Section 5.4, under which D still bounds the diameters; for the oracle, $\partial\vartheta_W$ has a closed graph, hence a Borel selection (Rockafellar and Wets, 1998, Ex. 14.9 and Cor. 14.6).

B.3 TFDP's generic cuts

TFDP (Forcier and Leclère, 2023, Algorithm 1) uses a γ^F -forward operator (their Def. 2.1), cuts of $\mathcal{B}_t$ applied to the already updated approximation (our $\underline{V}_{t+1}^k$) that are L_t -Lipschitz on the reachable set, valid and γ_t -tight (their Assumption 4(i)–(iii)), and a monotone update (their line 11). With TFDP's indexing, take X_t equal to their reachable set X_t^r , which is forward invariant by definition. Their Assumptions 1 and 3(i) make X_t^r bounded but not necessarily closed; we assume that it is closed, hence compact, as (S1) requires. TFDP starts from $V^0 \equiv -\infty$, so $M_t = \infty$; we start the analysis after the first iteration, conditionally on $\mathcal{A}_1$, with the first cuts $\underline{V}_t^1$ as initial approximations.

Proposition B.3.1 (TFDP with random node selection). *Consider TFDP's Algorithm 1 under their Assumptions 1–5, with closed reachable sets, with (S2) (e.g. losses bounded below and $\mathcal{B}_t(0)$ bounded), and with the noise of the forward pass drawn from the law of ξ_t , independently of the past, as in Forcier and Leclère (2023, Thm. 4.11). Take their constants: $n_t \leq n$, $D_t = D$, $L_t = \bar{L}_t = L$*

(Lipschitz constants of the cuts and of V_t), $\gamma_t \leq \gamma$. Then [Assumptions A](#) and [B](#) hold with $U = V$ from the second iteration on, with $\sup_{X_t}(V_t - \underline{V}_t^1) \leq \sum_{t \leq s < T}(\gamma_s + (\tilde{L}_s + L_s)D_s)$, and

$$\mathbb{E} \#\{k \geq 2 : G^k > \varepsilon + \Gamma\} \leq \frac{T(T+1)(\gamma + 2LD)(1 + 8LDT/\varepsilon)^n}{\varepsilon}, \quad \Gamma = \sum_{t < T} \gamma_t + \sum_{t \in [T]} \gamma_t^F. \quad (44)$$

Proof of [Proposition B.3.1](#). The setting satisfies [\(O1\)](#), [\(O2\)](#) ([Lemma 2.2\(b\)](#)), [\(O3\)](#) with $U_t = V_t$ ([Lemma 2.2\(a\)](#)) and [\(O4\)](#). Condition [\(S3\)](#) holds under TFDP's normal-integrand assumptions if the cut oracle is a Borel function of its inputs: the cuts are then Lipschitz and Borel in finitely many random parameters (the trial points and noises drawn so far), and V_{t+1} is Lipschitz ([Appendix A.4](#)). With random node selection from the law of ξ_t , the forward noise has the law used in the Bellman operators, even without finite support. Since V_t is $\tilde{L}_t$ -Lipschitz ([Forcier and Leclère, 2023](#), Assumption 3(ii)), we have, at $x = x_t^1$,

$$\sup_{X_t}(V_t - \underline{V}_t^1) \leq (V_t - \underline{V}_t^1)(x) + (\tilde{L}_t + L_t)D_t \leq \sup_{X_{t+1}}(V_{t+1} - \underline{V}_{t+1}^1) + \gamma_t + (\tilde{L}_t + L_t)D_t, \quad (45)$$

using $\underline{V}_t^1(x) \geq \mathcal{B}_t(\underline{V}_{t+1}^1)(x) - \gamma_t$ and $V_t(x) - \mathcal{B}_t(\underline{V}_{t+1}^1)(x) \leq \sup(V_{t+1} - \underline{V}_{t+1}^1)$. So $M_t^{(1)} := \sup(V_t - \underline{V}_t^1) \leq \sum_{t \leq s < T}(\gamma_s + (\tilde{L}_s + L_s)D_s)$, which is deterministic. The reachable sets need not be convex, so we apply [Theorem 3.5\(b\)](#) conditionally on $\mathcal{A}_1$, with nets of size $(1 + 8LDT/\varepsilon)^n$ ([Lemma A.1.1\(a\)](#)), $\sigma_t = \varepsilon/(4LT)$ and $\sum_{t < T} M_t^{(1)} \leq T(T+1)(\gamma + 2LD)/2$. $\square$

B.4 Reverse-norm and augmented Lagrangian cuts (SLDP)

SLDP uses the reverse-norm cut ([Ahmed et al., 2022](#), Def. 1) $f_t^k(y) := \mathcal{B}_t(\underline{V}_{t+1}^k)(x) - L_t\|y - x\|$ at $x = x_t^k$, averaged over the noise as on their p. 772; their ρ_t is our L_t . The state sets X_t , possibly nonconvex, are the compact projections of their Compact State Complete Continuous Recourse condition (p. 774).

Proposition B.4.1 (Reverse-norm cuts). *Let $\underline{V}_t^0$ be L_t -Lipschitz with $\underline{V}_t^0 \leq V_t$.*

- (i) [\(O1\)](#) holds with constant L_t , [\(O2\)](#) holds with $\gamma = 0$, and [\(O4\)](#) holds.
- (ii) *If $L_t \geq \text{Lip}_{X_t}(V_t)$ for every t , then every cut is below V_t on X_t , so [\(O3\)](#) holds with $U_t = V_t = \bar{U}_t$ and $M_t = \sup(V_t - \underline{V}_t^0)$.*

Proof of [Proposition B.4.1](#). (i) holds by construction. (ii) Induction: if $\underline{V}_{t+1}^k \leq V_{t+1}$ on X_{t+1} , then $\mathcal{B}_t(\underline{V}_{t+1}^k)(x) \leq V_t(x)$, so $f_t^k(y) \leq V_t(x) - L_t\|y - x\| \leq V_t(y)$; hence $\underline{V}_t^k \leq V_t$. $\square$

So [Theorem 3.5](#) only needs $L_t \geq \text{Lip } V_t$. SLDP's own validity condition ([Ahmed et al., 2022](#), eq. (20)), $f_t^k \leq \mathcal{B}_t(\underline{V}_{t+1}^k)$ for the *current* approximation, holds when $L_t \geq \text{Lip } \mathcal{B}_t(W)$ for every L_{t+1} -Lipschitz W , which their Hoffman recursion (17) guarantees. It can fail when L_t only dominates $\text{Lip } V_t$. Take identity dynamics on $X_1 = X_2 = [-1, 1]$, $T = 3$, $\mathcal{X}_1(x_0) = \{0\}$, zero losses, $L_2 = 1$, $\underline{V}_2^0(y) = \max(-1, -|y|)$ and $L_1 = 0$: the first cut of index 1, at $x_1^1 = 0$, is the constant $0 > -|y| = \mathcal{B}_1(\underline{V}_2^1)(y)$ for $y \neq 0$. Without snapping, [Theorem 3.5](#) gives explicit rates, without the error term of [Ahmed et al. \(2022, Lemma 5\)](#).

Remark B.4.2 (Augmented Lagrangian cuts). An augmented Lagrangian cut $g^{\text{AL}}(b_0; \lambda, \varrho) + \lambda^\top(b_0 - b) - \varrho\|b_0 - b\|$ of a nodal value function g at the trial state b_0 , with multiplier λ , penalty ϱ and augmented Lagrangian dual function g^{AL} ([Ahmed et al., 2022](#), Def. 3), is valid for g (weak duality), is $(\|\lambda\|_* + \varrho)$ -Lipschitz, and is tight if and only if $g^{\text{AL}}(b_0; \lambda, \varrho) = g(b_0)$. Uniformly Lipschitz cuts require restricting the dual to $\|\lambda\|_* + \varrho \leq L_t$. The restricted dual value is at least $\inf_b\{g(b) + L_t\|b - b_0\|\}$, which equals $g(b_0)$ when $L_t \geq \text{Lip}(g)$ on the whole domain of g ([Ahmed et al., 2022](#), p. 768); the

cut then satisfies [Assumption B](#) with $\gamma = 0$ and $U = V$. Otherwise the cut is tight only for a Lipschitz-regularised operator, the setting of [Section 4.2](#) with a norm regularisation. With SLDP’s unregularised forward pass, (O2) is then guaranteed only with γ_t equal to a regularisation gap, which is not controlled a priori. [Füllner et al. \(2024\)](#) study Lipschitz regularisation and the tightness of Lagrangian cuts in this setting.

B.5 Zhang and Sun

Setting and dictionary. [Zhang and Sun \(2022\)](#) work on a recombining tree (their Assumption 3) with stages $0 \leq t \leq T - 1$, compact, possibly nonconvex state spaces and nonnegative lower semicontinuous nodal costs (possibly $+\infty$, which we encode in the feasible sets). Their stage t is our stage $t + 1$, and their root is our stage 1. Their state space at stage t , written $\mathcal{S}_t$ here, is our X_{t+1} ; their approximations $\mathcal{Q}_t^i$ and original expected cost-to-go functions $\mathcal{Q}_t$ (their (3)) are our $\underline{V}_{t+1}^i$ and V_{t+1} . Within a stage we use their node notation: n is a node (not a dimension), m ranges over its children, p_{nm} are the transition probabilities, F_m the nodal feasible sets, f_m the nodal costs and $\mathcal{Q}_m^i$ the approximations; $\mathcal{R}_n$, ϖ_n and ϱ are their ψ_n , σ_n and ρ . Their regularised value functions $\mathcal{Q}_n^R$ (their (4)) are our V^R , the value functions of the regularised operator $\mathcal{B}_t^R$, with stage cost $\min_{z,y} \{f(z, y, x) + \varpi \mathcal{R}(\bar{x} - z)\}$ at the incoming state $\bar{x}$ and next-state set $\text{proj}_x F_t$. The forward step of their Algorithm 3 (line 7) is greedy for $\mathcal{B}_t^R$. The backward step (line 14) returns the generalised conjugacy cut C_m^i of each child m , with constant v_m^i (the value of their problem (B) at the parent state) and parameters (λ, ϱ) bounded by $l_{n,\lambda}$ and $l_{n,\varrho}$ (their (15)–(16)); the update is their (20). Exact penalisation (their Assumption 2) means $v^{\text{reg}} = v^{\text{prim}}$, and a root solution is ε -optimal (their Def. 1) if its regularised cost is at most $v^{\text{reg}} + \varepsilon$.

Proposition B.5.1 (Zhang–Sun cuts satisfy the oracle). *Under Assumptions 1 and 3 of [Zhang and Sun \(2022\)](#), with their parameter bounds $l_{n,\varrho} \geq \varpi_n$, or, in the convex case, $\mathcal{R}_n = \|\cdot\|$, $l_{n,\lambda} \geq \varpi_n$ and $l_{n,\varrho} = 0$, Algorithm 3 with $\mathcal{B}_t^R$ in the role of $\mathcal{B}_t$ satisfies [Assumptions A](#) and [B](#) with $\text{dist}_t = \|\cdot\|$; L equal to their constants $L_n = \sum_m p_{nm}(l_{m,\lambda} + l_{m,\varrho})$ ([Zhang and Sun, 2022, Prop. 5](#)); $\gamma = \gamma^F = 0$; and $U = V$, the value functions of the original problem.*

Lemma B.5.2 (Validity for the regularised value functions). *If moreover $\mathcal{R}_m$ is subadditive (e.g. a norm) and $l_{m,\lambda} + l_{m,\varrho} \leq \varpi_m$ (with the bounds of [Proposition B.5.1](#), this means $(l_{m,\lambda}, l_{m,\varrho}) = (0, \varpi_m)$, or $(\varpi_m, 0)$ in the convex case), then the cuts are also valid for the regularised functions, $\mathcal{Q}_n^i \leq \mathcal{Q}_n^R$, and U can be taken equal to V^R .*

Proof of [Proposition B.5.1](#). (S1): the regularised next-state set $\text{proj}_x F_t \subset \mathcal{S}_t$ is nonempty and does not depend on the incoming state. (S2): $f \geq 0$, so (S2) reduces to the boundedness of $\mathcal{B}_t^R(0)$, and $\mathcal{B}_t^R(0)(\bar{x}) \leq \min f + \varpi \sup_{\mathcal{S}-\mathcal{S}} \mathcal{R} < \infty$ uniformly in $\bar{x}$, because Assumption 1 makes the problem proper. (S3): the oracles return minimisers, and the tree is finite. (O1) and (O3): by Prop. 5 of [Zhang and Sun \(2022\)](#), $\mathcal{Q}_n^i$ is L_n -Lipschitz and lies below $\mathcal{Q}_n$ on $\mathcal{S}_n$. Its proof shows the same for each averaged cut $\sum_m p_{nm} C_m^i$, which lies below $\sum_m p_{nm} \mathcal{Q}_m = \mathcal{Q}_n$ by their (12), (16) and (B). (O2): the first display chain of the proof of their Lemma 3 (their App. A.3.1) gives, for each child m , at the parent state $\bar{x}$,

$$v_m^i \geq \min_{(x,y) \in F_m, z \in \mathcal{S}_n} f_m(z, y, x) + \varpi_m \mathcal{R}_m(\bar{x} - z) + \mathcal{Q}_m^i(x). \quad (46)$$

In the nonconvex case, $(\lambda, \varrho) = (0, \varpi_m)$ is feasible in their (B); in the convex case, use their Lemma 2 and $l_{m,\lambda} \geq \varpi_m$. Averaging over m , the averaged cut at $\bar{x}$ is at least $\mathcal{B}^R(\mathcal{Q}^i)(\bar{x}) \geq \mathcal{B}^R(\mathcal{Q}^{i-1})(\bar{x})$. The root step (lines 19–20) is exact. (O4) is their (20). $\square$

Proof of Lemma B.5.2. Let $G_m(x) := \min_{(x', y) \in F_m} f_m(x, y, x') + Q_m^i(x')$; by weak duality, $C_m^i \leq G_m$ on $\mathcal{S}_n$ (Zhang and Sun, 2022, proof of Prop. 5). For $x, z \in \mathcal{S}_n$, $C_m^i(x) - C_m^i(z) = \langle \hat{\lambda}, x - z \rangle + \hat{\rho}[\mathcal{R}_m(\bar{x} - z) - \mathcal{R}_m(\bar{x} - x)] \leq \varpi_m \mathcal{R}_m(x - z)$: in the case $(0, \varpi_m)$, $\hat{\lambda} = 0$, $\hat{\rho} \leq \varpi_m$ and $\mathcal{R}_m$ is subadditive; in the case $(\varpi_m, 0)$, $\hat{\rho} = 0$, $\|\hat{\lambda}\|_* \leq \varpi_m$ and $\mathcal{R}_m = \|\cdot\|$. Hence $C_m^i(x) \leq \inf_z \{G_m(z) + \varpi_m \mathcal{R}_m(x - z)\}$, the regularised nodal value at x with approximation Q_m^i . Averaging over m gives $\leq \mathcal{B}^R(Q^i)(x) \leq \mathcal{B}^R(Q^R)(x) = Q_n^R(x)$, by backward induction over the tree and induction over i (with $Q^0 = 0 \leq Q^R$). $\square$

Proposition B.5.3 (General form of Theorem 4.3). *In the setting of Theorem 4.3, without the assumptions on Lipschitz constants, diameters and costs, let L_s be the Lipschitz constant of the approximations of index s (Proposition B.5.1), let $X_s \subset \bigcup_{j \leq J_s} B_{s,j}$ with balls of diameters $D_{s,j}$, and let $I_s(\varepsilon_s) := \sum_{j \leq J_s} (1 + 2L_s D_{s,j}/\varepsilon_s)^{d_s}$. Put $M_0 := v^{\text{prim}}$, the optimal value, and $M_s := \sup_{x \in X_s} \inf_{z \in X_s} \{V_s(z) + L_s \|x - z\|\}$. Then, for all $(\varepsilon_s)_{1 \leq s < T} \subset (0, \infty)$ with $\sum_s \varepsilon_s \leq \varepsilon/4$,*

$$\mathbb{E}[\iota] \leq \frac{2}{\varepsilon} \left(M_0 + \sum_{s=1}^{T-1} I_s(\varepsilon_s) M_s \right), \quad (47)$$

and, for $\delta \in (0, 1]$, with probability at least $1 - \delta$, $\iota \leq \frac{18 \ln(3/\delta)}{\varepsilon} (M_0 + \sum_s I_s(\varepsilon_s) M_s)$. The same bounds hold for the number of iterations at which the regularised policy is not ε -optimal for the regularised problem, or the lower bound is not ε -tight.

The bound has the structure of the deterministic Theorem 2 of Zhang and Sun (2022), $i \leq \sum_s I_s(\varepsilon_s)$ with $\sum_s \varepsilon_s = \varepsilon$ (our net count is their ball count), with an extra factor $2M_s/\varepsilon$ and tolerances smaller by a factor 4; their random-sampling bound has an extra factor of order $\prod_s N_s$.

Proof of Theorem 4.3 and Proposition B.5.3. Zhang and Sun's oracles are random functions measurable with respect to σ -fields Σ_i^{oracle} . We assume the sampled paths independent of $\Sigma_\infty^{\text{oracle}}$ and include $\Sigma_\infty^{\text{oracle}}$ in $\mathcal{A}_0$, so that the oracle randomness meets the requirement of Section 2.2. By Proposition B.5.1, Theorem 3.5(b) applies to the regularised problem, with G^k the regularised policy cost minus the lower bound, also with several paths per iteration (Section 3.4). By their Assumption 2, $U_0(x_0) = v^{\text{prim}} = v^{\text{reg}} = V_0^R(x_0)$, so $\zeta_0 = 0$. Hence $G^k \leq \varepsilon$ implies: the regularised policy has cost at most $v^{\text{reg}} + \varepsilon$; its root decision satisfies their Def. 1; and $v^{\text{prim}} - V_0^{k-1}(x_0) \leq \varepsilon$. So $\iota \leq \min\{k \geq 1 : G^k \leq \varepsilon\} - 1 \leq \#\{k \geq 1 : G^k > \varepsilon\}$ (their iteration $i + 1$ is our iteration $k = i + 1$). Take the net $\{x_0\}$ of X_0 , with $M_0 = U_0(x_0) = v^{\text{prim}}$ since $V^0 = 0$, and, for $1 \leq s < T$, $\sigma_s := \varepsilon_s/L_s$ if $L_s > 0$ and $\sigma_s := \max_j D_{s,j}$ if $L_s = 0$, so that $2 \sum_{s < T} L_s \sigma_s \leq 2 \sum_s \varepsilon_s \leq \varepsilon/2$. If $L_s > 0$, a maximal σ_s -separated subset of $X_s \cap B_{s,j}$ has at most $(1 + 2D_{s,j}/\sigma_s)^{d_s}$ points, by the packing argument of Lemma A.1.1(a); if $L_s = 0$, one point of each nonempty $X_s \cap B_{s,j}$ suffices. In both cases X_s has a σ_s -net of size at most $I_s(\varepsilon_s)$, and M_s is the potential bound of (O3) with $U = V$ (under Lemma B.5.2, it may be replaced by $\sup_{X_s} V_s^R$). This proves Proposition B.5.3. For the explicit bounds, $M_0 \leq T\bar{\ell}$ and $\sum_{1 \leq s < T} M_s \leq \bar{\ell} T(T-1)/2$, so, at an accuracy θ and with $I := \max_s I_s(\varepsilon_s)$, $\mathbb{E}[\iota] \leq (2/\theta)(T\bar{\ell} + I\bar{\ell} T(T-1)/2) \leq I\bar{\ell} T(T+1)/\theta$. For ε -optimality, take $\theta = \varepsilon$ and $\varepsilon_s = \varepsilon/(4(T-1))$, so $I = (1 + 8LD(T-1)/\varepsilon)^d$; this is (25). For the accuracy $T\varepsilon$ of Table 1, take $\theta = T\varepsilon$ and $\varepsilon_s = T\varepsilon/(4(T-1)) \geq \varepsilon/4$, so $I \leq (1 + 8LD/\varepsilon)^d$ and $\mathbb{E}[\iota] \leq \bar{\ell}(T+1)(1 + 8LD/\varepsilon)^d/\varepsilon$, linear in T . $\square$

Remark B.5.4 (Uniform sampling on non-uniform trees). If the forward noise is drawn from q_t instead of the transition law p_t , with $p_t \leq v_t q_t$, the lifts are obtained along q -paths while G^k is a p -expectation. Since $(r_t^k)_+ \geq 0$, the expectation under p of the positive residuals is at most $\Upsilon := \prod_{s < T-1} v_s$ times that under q , and the proof of Theorem 3.5 gives $\mathbb{E} \sum_k \Psi_T(G^k/\Upsilon) \leq \sum_{t < T} M_t$

(exact cuts). Under the uniform Assumption 4 of Zhang and Sun, $v_s = N_s \max p_{nm}$ over the nodes n of stage s , each with N_s children m , so Υ is exponential in T unless the transitions are nearly uniform.

B.6 SDDiP

Proof of Proposition 4.4. The validity condition of Zou et al. (2019, (3.4)) is validity for the *true* value function on $\{0, 1\}^{n_t}$, which is (O3) with $U_t = V_t$. Their tightness (3.5) is tightness for the nodal problem with the updated approximation, which is (O2) with $\gamma = 0$ by Lemma 2.2(b). Their Theorem 3 proves both properties for Lagrangian cuts; for their integer optimality cut (4.2) both are immediate. Additional valid cuts satisfy (O3). On a finite set, the Lipschitz part of (O1) is not needed (Theorem 3.5(c)), $\underline{V}^0 = 0$ is deterministic, and $M_t \leq (T - t)\bar{\ell}$. $\square$

B.7 MIDAS step cuts

Step cuts are the one family in this paper that is not Lipschitz: this subsection shows that (O1) is not necessary, since monotone approximations can replace Lipschitz ones in the volume argument. MIDAS (Philpott et al., 2020), written here for minimisation, uses step cuts $f(y) = v \mathbf{1}\{y < x + \omega \mathbf{1}\}$ (componentwise inequality), with $v = \mathcal{B}_t(\underline{V}_{t+1}^k)(x)$ at the trial point x and step width $\omega > 0$ (their δ). Every $\underline{V}_t^k$ is then nonincreasing, but (O1) and Lemma 3.3 fail. Take $X_t = [0, 1]$, $\underline{V}_t^{k-1} = v' \mathbf{1}\{y < 1 - \omega'\}$ with $v' \geq v$ and $\omega' > 0$, and the trial point $x = 1$ (at distance $\omega + \omega'$ from the earlier trial point $1 - \omega' - \omega$): the new cut equals v on X_t , but the lift is positive only on $[1 - \omega', 1]$, whose measure ω' is arbitrary. Without snapping (Section 4.4), monotonicity of the approximations can replace their Lipschitz continuity (Theorem B.7.1; MIDAS's own guarantee is asymptotic, with floor $(T + 1)\varepsilon$). TFDP's suggestion (Forcier and Leclère, 2023, App. A.3) to interpolate affinely between the steps gives cuts that are Lipschitz in ℓ_∞ with constant of order $(T - t)\bar{\ell}/\omega$, which Theorem 3.5(b) covers.

Theorem B.7.1 (Monotone step cuts). *Let Assumption A hold, with losses in $[0, \bar{\ell}]$ and $\underline{V}_t^0 \equiv 0$ for $t < T$. Fix $\omega > 0$ and let $X_t^+ := X_t + [0, \omega]^{n_t}$. Let every cut be a step cut $f(y) = v \mathbf{1}\{y < x + \omega \mathbf{1}\}$ as in Appendix B.7, extended to $\mathbb{R}^{n_t}$ by the same formula, with $\mathcal{B}_t(\underline{V}_{t+1}^{k-1})(x) - \gamma_t \leq v \leq \mathcal{B}_t(\underline{V}_{t+1}^k)(x)$ at $x = \mathbf{x}_t^k$, and let the update be monotone. Let the forward pass be as in Section 2.2, with no snapping. Then*

$$\mathbb{E} \sum_{k \geq 1} (G^k - \Gamma)_+ \leq \bar{\ell} \sum_{t < T} (T - t) \frac{\text{vol}(X_t^+)}{\omega^{n_t}} \leq \bar{\ell} \frac{T(T + 1)}{2} \left(1 + \frac{D}{\omega}\right)^n, \quad (48)$$

where the last bound holds if each X_t lies in a box of side D and $n_t \leq n$. If moreover $V_t(y) \geq V_t(x) - \varepsilon$ whenever $x, y \in X_t$ and $y < x + \omega \mathbf{1}$ (the minimisation form of the approximate monotonicity condition (50) of Philpott et al., 2020), then $\underline{V}_t^k \leq V_t + (T - t)\varepsilon$ on X_t , so $V_0^{\pi^k}(x_0) - V_0(x_0) \leq G^k + T\varepsilon$.

Proof of Theorem B.7.1. Since $\underline{V}^0 = 0$, losses are nonnegative and $v \leq \mathcal{B}_t(\underline{V}_{t+1}^k)(x) \leq \bar{\ell} + \sup \underline{V}_{t+1}^k$, backward induction gives $0 \leq \underline{V}_t^k \leq (T - t)\bar{\ell}$ on $\mathbb{R}^{n_t}$. Lemma 3.2 uses nothing beyond boundedness, so it holds. Fix $k \geq 1$, $t < T$, $x = \mathbf{x}_t^k$ and $r_t = r_t^k(x)$. On the box $Q := \{y : x \leq y < x + \omega \mathbf{1}\} \subset X_t^+$, $f_t^k(y) = v$ and $\underline{V}_t^{k-1}(y) \leq \underline{V}_t^{k-1}(x)$, because $\underline{V}_t^{k-1}$ is nonincreasing. Hence $\underline{V}_t^k(y) - \underline{V}_t^{k-1}(y) \geq v - \underline{V}_t^{k-1}(x) \geq (r_t - \gamma_t)_+$ on Q , using $\underline{V}_t^k \geq \underline{V}_t^{k-1}$. Since $\text{vol}(Q) = \omega^{n_t}$, the potential $\Phi^k := \sum_{t < T} \int_{X_t^+} \underline{V}_t^k$ satisfies $\Phi^k - \Phi^{k-1} \geq \sum_{t < T} \omega^{n_t} (r_t - \gamma_t)_+$ and $\Phi^k \leq \bar{\ell} \sum_{t < T} (T - t) \text{vol}(X_t^+)$. Normalise stage t by ω^{n_t} and proceed as in the proof of Theorem 3.5(c). For the last claim, let $\zeta_t := \sup_k \sup_{X_t} (\underline{V}_t^k - V_t)_+$. Then $v \leq \mathcal{B}_t(V_{t+1} + \zeta_{t+1})(x) = V_t(x) + \zeta_{t+1}$. If $y \in X_t$ and $y < x + \omega \mathbf{1}$, then $V_t(y) \geq V_t(x) - \varepsilon$, so $f(y) \leq V_t(y) + \varepsilon + \zeta_{t+1}$ for a step cut f ; elsewhere $f = 0 \leq V_t$. Hence

$\zeta_t \leq \varepsilon + \zeta_{t+1}$, and $\zeta_T = 0$; this is the minimisation form of [Philpott et al. \(2020, Lemma 9\)](#). Finally, [Lemma 3.2](#) gives $V_0^{\pi^k}(x_0) - V_0(x_0) \leq G^k + \zeta_0$. $\square$

C Proofs and complements for Section 5

C.1 Exact cuts for the true law

Write E_t for the direction space of $\text{aff } X_t$.

Proposition C.1.1 (Expected Benders cuts). *Let W be continuous on X_{t+1} and $x \in X_t$. Suppose there are Borel functions $a : \Xi_{t+1} \rightarrow \mathbb{R}$ and $g : \Xi_{t+1} \rightarrow E_t$ with $a(\xi) = \dot{\mathcal{B}}_t(W; x, \xi)$, $\|g(\xi)\| \leq L_t$ and $\dot{\mathcal{B}}_t(W; y, \xi) \geq a(\xi) + \langle g(\xi), y - x \rangle$ for all $y \in X_t$ and $\xi \in \Xi_{t+1}$. Then $f(y) := \mathbb{E}a(\xi_{t+1}) + \langle \mathbb{E}g(\xi_{t+1}), y - x \rangle$ is an L_t -Lipschitz affine minorant of $\mathcal{B}_t(W)$ on X_t with $f(x) = \mathcal{B}_t(W)(x)$.*

Proof. Integrate the pointwise inequality against $\mathbb{P}_{t+1}$; it is an equality at $y = x$, and $\|\mathbb{E}g\| \leq L_t$. $\square$

Remark C.1.2 (Linear stages). For linear stages with finite optimal values, data affine in ξ , a deterministic full-row-rank recourse matrix and a density of $\mathbb{P}_{t+1}$ that is polynomial on each cell of a polyhedral subdivision of a polytope Ξ_{t+1} , the expectations of [Proposition C.1.1](#) are finite sums of integrals of polynomials over polytopes, computable exactly for rational data ([Baldoni et al., 2011](#)). Indeed, for fixed x , the noises for which a given basis is optimal form a polytope, on which the value and the subgradient given by the dual solution are polynomials of degree at most two in ξ ; conclude by inclusion–exclusion over these polytopes and the density cells. TFDP ([Forcier and Leclère, 2023, Thm. 3.7](#)) builds a stronger exact cut in a related setting.

C.2 Jensen lower discretisation

For $t+1 < T$ and convex X_{t+1} , let $\mathcal{W}_{t+1}^c$ be the set of convex L_{t+1} -Lipschitz functions $W : X_{t+1} \rightarrow \mathbb{R}$ with $\underline{V}_{t+1}^0 \leq W \leq V_{t+1}$, and let $\mathcal{W}_T^c := \{0\}$. If $\underline{V}_{t+1}^0 \leq V_{t+1}$ is convex and L_{t+1} -Lipschitz, and every cut of index $t+1$ is affine, L_{t+1} -Lipschitz and valid, then every $\underline{V}_{t+1}^k$ lies in $\mathcal{W}_{t+1}^c$. The conditions of [Proposition C.2.1](#) and [Assumption F](#) are imposed on this class only, since a nonconvex W need not have tight affine minorants. The classes $\mathcal{W}_{t+1}$ of (27) and $\mathcal{W}'_{t+1}$ of [Proposition C.3.1](#) need no convexity: they only enter suprema that bound biases.

Proposition C.2.1 (Jensen lower discretisation). *Fix $t < T$. Assume that X_t, X_{t+1} and $\Xi_{t+1} \subset \mathbb{R}^q$ are convex, that $\mathbb{P}_{t+1}$ has a finite first moment, and that, for every $W \in \mathcal{W}_{t+1}^c$ and every $y \in X_t$, the map $\xi \mapsto \dot{\mathcal{B}}_t(W; y, \xi)$ is convex on Ξ_{t+1} and L_t^ξ -Lipschitz. Let $\mathcal{P}$ be a finite partition of Ξ_{t+1} into convex Borel cells C , $p_C := \mathbb{P}_{t+1}(C)$ and, if $p_C > 0$, $\xi_C := \mathbb{E}[\xi_{t+1} \mid \xi_{t+1} \in C]$. For such a W and $x \in X_t$, let f_C be an affine L_t -Lipschitz minorant of $\dot{\mathcal{B}}_t(W; \cdot, \xi_C)$ on X_t with $f_C(x) = \dot{\mathcal{B}}_t(W; x, \xi_C)$. Then $f := \sum_{C: p_C > 0} p_C f_C$ is affine, L_t -Lipschitz, valid for $\mathcal{B}_t(W)$ on X_t and γ -tight at x with $\gamma \leq L_t^\xi e(\mathcal{P})$, where $e(\mathcal{P}) := \mathbb{E}\|\xi_{t+1} - \mathbb{E}[\xi_{t+1} \mid \mathcal{P}]\| \leq \max_{C \in \mathcal{P}} \text{diam } C$. If Ξ_{t+1} lies in a cube Q of side $D_\infty > 0$ and, for $h > 0$ and $m := \lceil D_\infty/h \rceil$, $\mathcal{P}$ consists of the nonempty traces on Ξ_{t+1} of the m^q boxes obtained by partitioning each side of Q into m intervals of length at most h , then $|\mathcal{P}| \leq m^q$ and, for the Euclidean norm, $e(\mathcal{P}) \leq h\sqrt{q}/2$.*

Proof. A probability law carried by a convex set C has its mean in C , and Jensen’s inequality holds for convex functions on C . Cells with $p_C = 0$ do not contribute to $\mathcal{B}_t(W)$; below, sums run over the cells with $p_C > 0$, whose weights add up to one, so f is affine and L_t -Lipschitz. For $y \in X_t$, $\dot{\mathcal{B}}_t(W; y, \xi_C) \leq \mathbb{E}[\dot{\mathcal{B}}_t(W; y, \xi_{t+1}) \mid \xi_{t+1} \in C]$, and $f(y) \leq \sum_C p_C \dot{\mathcal{B}}_t(W; y, \xi_C) \leq \mathcal{B}_t(W)(y)$. At x , $\mathcal{B}_t(W)(x) - f(x) = \sum_C p_C (\mathbb{E}[\dot{\mathcal{B}}_t(W; x, \xi_{t+1}) \mid C] - \dot{\mathcal{B}}_t(W; x, \xi_C)) \leq L_t^\xi \sum_C p_C \mathbb{E}[\|\xi_{t+1} - \xi_C\| \mid C] =$

$L_t^\xi e(\mathcal{P})$. For boxes of sides at most h , each coordinate of ξ_{t+1} lies, given the cell, in an interval of length at most h , so its conditional variance is at most $h^2/4$, and $e(\mathcal{P}) \leq (\mathbb{E}\|\xi_{t+1} - \mathbb{E}[\xi_{t+1} | \mathcal{P}]\|^2)^{1/2} \leq h\sqrt{q}/2$. $\square$

An arbitrary grid of half-open cubes of side h gives $|\mathcal{P}| \leq (m+1)^q$. For $W \in \mathcal{W}_{t+1}^c$, convexity in ξ holds when the noise enters the right-hand side or the technology matrix of a linear stage; it can fail for a nonconvex W (if the next state equals the noise, $\dot{\mathcal{B}}_t(W; y, \xi) = \ell_{t+1}(y, \xi, \xi) + W(\xi)$). When the noise enters the cost, Edmundson–Madansky weights give the lower bound instead (Birge and Wets, 1986; Kuhn, 2005; Forcier and Leclère, 2023, Prop. 2.6).

C.3 SAA backward passes: proofs

End of the proof of Theorem 5.1. By Lemma 3.1(iii), $V_0^{\pi^k}(x_0) \geq V_0(x_0)$, the first inequality of (30). Moreover $\underline{V}_0^{k-1}(x_0) \leq \widehat{V}_0(x_0)$, and by backward induction $\sup_{X_t} (\widehat{V}_t - V_t)_+ \leq \sum_{t \leq s < T} (\Delta_s^-)_+$. Indeed $\widehat{V}_t - V_t = [\widehat{\mathcal{B}}_t(\widehat{V}_{t+1}) - \widehat{\mathcal{B}}_t(V_{t+1})] + [\widehat{\mathcal{B}}_t(V_{t+1}) - \mathcal{B}_t(V_{t+1})]$, and the first bracket is at most $\sup_{X_{t+1}} (\widehat{V}_{t+1} - V_{t+1})_+$, because $\widehat{\mathcal{B}}_t$ is monotone, commutes with constants, and $\mathcal{X}_{t+1}(x, \xi) \subset X_{t+1}$. Hence $G^k = V_0^{\pi^k}(x_0) - \underline{V}_0^{k-1}(x_0) \geq V_0^{\pi^k}(x_0) - V_0(x_0) - \sum_{t < T} (\Delta_t^-)_+$. $\square$

Proof of Corollary 5.2. (a) For a coupling $(\xi, \hat{\xi})$ of $(\mathbb{P}_{t+1}, \widehat{\mathbb{P}}_{t+1})$ on Ξ_{t+1}^2 and $f = \dot{\mathcal{B}}_t(W; x, \cdot)$, $|\int f d\mathbb{P}_{t+1} - \int f d\widehat{\mathbb{P}}_{t+1}| = |\mathbb{E}[f(\xi) - f(\hat{\xi})]| \leq L_t^\xi \mathbb{E}\|\xi - \hat{\xi}\|$; take the infimum over couplings. (b) With $\underline{f} := \inf f$, $f - \underline{f} \in [0, \text{rg}_t]$, so $\int (f - \underline{f}) d(\mathbb{P}_{t+1} - \widehat{\mathbb{P}}_{t+1}) \leq \text{rg}_t \sup_A (\mathbb{P}_{t+1} - \widehat{\mathbb{P}}_{t+1})(A)$, and symmetrically. $\square$

When the bias of Corollary 5.2(a) is small. Take additive noise in the dynamics, $\mathcal{X}_{t+1}(x, \xi) = \{A_{t+1}x + u + \xi : u \in \mathcal{U}_{t+1}\}$ with $\mathcal{U}_{t+1}$ compact, and a loss such that $\xi \mapsto \ell_{t+1}(x, A_{t+1}x + u + \xi, \xi)$ is L_{t+1}^ℓ -Lipschitz for all $x \in X_t$ and $u \in \mathcal{U}_{t+1}$ (e.g. $L^{\ell, y}$ -Lipschitz in the next state and $L^{\ell, \xi}$ -Lipschitz in ξ , with $L_{t+1}^\ell = L^{\ell, y} + L^{\ell, \xi}$). If X_{t+1} is forward invariant for $\xi \in \Xi_{t+1}$, and V_{t+1} and the elements of $\mathcal{W}_{t+1}$ are L_{t+1} -Lipschitz for the norm of Ξ_{t+1} , the Lipschitz condition holds with $L_t^\xi \leq L_{t+1}^\ell + L_{t+1}$. Let $\widehat{\mathbb{P}}_t$ be the empirical law of N i.i.d. draws from $\mathbb{P}_t$, with $\text{diam } \Xi_t \leq D_\xi$. Replacing one sample point moves a mass $1/N$ by at most D_ξ , so it changes $\mathbb{W}_1(\mathbb{P}_t, \widehat{\mathbb{P}}_t)$ by at most D_ξ/N . McDiarmid’s inequality (McDiarmid, 1989) with a union bound over t then gives, with probability at least $1 - \delta$, $\mathbb{W}_1(\mathbb{P}_t, \widehat{\mathbb{P}}_t) \leq \mathbb{E}\mathbb{W}_1(\mathbb{P}_t, \widehat{\mathbb{P}}_t) + D_\xi \sqrt{\ln(T/\delta)/(2N)}$ for all $t \in [T]$; the mean is bounded in Section 5.3.

Proposition C.3.1 (i.i.d. SAA: a bias governed by the state dimension). *Let $\|\cdot\|$ be a norm, for which L_{t+1} , L_t^x and the diameters are taken. For each $t < T$, let $\mathcal{W}'_{t+1}$ be the set of L_{t+1} -Lipschitz W on X_{t+1} (set $L_T = D_T := 0$, so that the terms of index T vanish) with $\underline{w}_{t+1} \leq W \leq \overline{w}_{t+1} + B_{t+1}$, and assume that it contains $\mathcal{W}_{t+1} \cup \{V_{t+1}\}$. Assume that, for every $\sigma > 0$, X_s has a σ -net for $\|\cdot\|$ of size at most $(1 + 2D_s/\sigma)^d$ for $s = t$ and, if $t + 1 < T$, for $s = t + 1$ (see Lemma A.1.1(a)). Assume that, for all $x, x' \in X_t$, $\xi \in \Xi_{t+1}$ and $W \in \mathcal{W}'_{t+1}$, $\dot{\mathcal{B}}_t(W; x, \cdot)$ takes values in an interval of length rg_t and $|\dot{\mathcal{B}}_t(W; x, \xi) - \dot{\mathcal{B}}_t(W; x', \xi)| \leq L_t^x \|x - x'\|$. Let $\widehat{\mathbb{P}}_{t+1}$ be the empirical law of N i.i.d. draws from $\mathbb{P}_{t+1}$. Then for every $u > 0$ and $\delta \in (0, 1)$, with probability at least $1 - \delta$, for all $t < T$,*

$$\sup_{x \in X_t, W \in \mathcal{W}'_{t+1}} |\mathcal{B}_t(W)(x) - \widehat{\mathcal{B}}_t(W)(x)| \leq 4u + \text{rg}_t \sqrt{\frac{\ln \nu_t(u) + \ln(2T/\delta)}{2N}}, \quad (49)$$

$$\ln \nu_t(u) := d \ln \left(1 + \frac{2L_t^x D_t}{u}\right) + \left(1 + \frac{8L_{t+1} D_{t+1}}{u}\right)^d \ln \left(1 + \frac{2B_{t+1}}{u}\right). \quad (50)$$

With $u = N^{-1/(d+2)}$, $\Delta_t^\pm = O(N^{-1/(d+2)} \sqrt{\ln N})$, with constants depending on d , $L_{t+1} D_{t+1}$, $L_t^x D_t$, B_{t+1} , rg_t and $\ln(T/\delta)$, but not on q .

With losses in $[0, \bar{\ell}]$ and $\underline{V}^0 = 0$, one can take $\underline{w} = 0$ and $B_{t+1} = (T - t - 1)\bar{\ell}$ if V_{t+1} is L_{t+1} -Lipschitz.

Proof. Nets. By assumption, take a (u/L_t^x) -net $\mathcal{N}$ of X_t and a $(u/(4L_{t+1}))$ -net $\{z_j\}_{j \in [n_Z]}$ of X_{t+1} , with $n_Z \leq (1 + 8L_{t+1}D_{t+1}/u)^d$ (a radius $u/0$ is infinite, and one point is then a net; this covers $t = T - 1$, where $L_T = 0$). *Cover of $\mathcal{W}'_{t+1}$.* Code W by the integers $\lfloor (W(z_j) - \underline{w})/(u/2) \rfloor \in \{m \in \mathbb{N} : m \leq \lfloor 2B/u \rfloor\}$. If W, W' have the same code, then for $y \in X_{t+1}$ and z_j with $\|y - z_j\| \leq u/(4L_{t+1})$, $|W(y) - W'(y)| \leq |W(y) - W(z_j)| + |W(z_j) - W'(z_j)| + |W'(z_j) - W'(y)| < u$. One member of $\mathcal{W}'_{t+1}$ in each nonempty code class gives a u -cover $\mathcal{W}_u$ in sup norm with $|\mathcal{W}_u| \leq (1 + 2B/u)^{n_Z}$. *Concentration.* For $(x', W') \in \mathcal{N} \times \mathcal{W}_u$, $f = \dot{B}_t(W'; x', \cdot)$ takes values in an interval of length rg_t ; Hoeffding's inequality (Hoeffding, 1963) and a union bound over these $\nu_t(u) \geq |\mathcal{N}||\mathcal{W}_u|$ functions and over t give, with probability at least $1 - \delta$, $|\int f d(\mathbb{P}_{t+1} - \hat{\mathbb{P}}_{t+1})| \leq \text{rg}_t \sqrt{(\ln \nu_t(u) + \ln(2T/\delta))/(2N)}$ for all of them. *Approximation.* For (x, W) , pick $x' \in \mathcal{N}$ with $\|x - x'\| \leq u/L_t^x$ and $W' \in \mathcal{W}_u$ with $\|W - W'\|_\infty \leq u$. Since $\mathcal{X}_{t+1}(x', \xi) \subset X_{t+1}$, $|\dot{B}_t(W; x', \xi) - \dot{B}_t(W'; x', \xi)| \leq u$, so $|\dot{B}_t(W; x, \xi) - \dot{B}_t(W'; x', \xi)| \leq 2u$ for every $\xi \in \Xi_{t+1}$, and both integrals move by at most $2u$. For the rate, $\ln \nu_t(u) = O(u^{-d} \ln(1/u))$, and $u = N^{-1/(d+2)}$ balances the two terms. $\square$

Remark C.3.2 (A faster rate for Δ_t^-). Only Δ_t^+ needs the cover of $\mathcal{W}'_{t+1}$. The bias Δ_t^- involves the single function V_{t+1} , which does not depend on the sample. Under the assumptions of Proposition C.3.1, Hoeffding's inequality on a $(D_t/\sqrt{N})$ -net of X_t , of size at most $(1 + 2\sqrt{N})^d$ (one point if $D_t = 0$), a union bound over the net and over t , and the L_t^x -Lipschitz continuity of $\mathcal{B}_t(V_{t+1})$ and $\hat{\mathcal{B}}_t(V_{t+1})$ give, with probability at least $1 - \delta$, for all $t < T$,

$$\Delta_t^- \leq \frac{2L_t^x D_t}{\sqrt{N}} + \text{rg}_t \sqrt{\frac{d \ln(1 + 2\sqrt{N}) + \ln(2T/\delta)}{2N}} = O\left(\sqrt{\frac{d \ln N}{N}}\right),$$

with constants depending on $L_t^x D_t$, rg_t and $\ln(T/\delta)$. The policy bias in (30) thus decreases at the parametric rate up to a logarithm; only the lift bias Δ_t^+ has the dimension d in the exponent.

Remark C.3.3 (Bias, and SAA forward passes). In the instance of Proposition C.4.3 (Appendix C.4), with a fixed SAA whose mean exceeds 0.6, SDDP ends with a policy that is 0.1-suboptimal for the true problem, with forward passes from either law; this is consistent with Theorem 5.1, since $\Delta_1^- > 0.1$. For the Jensen law $\sum_C p_C \delta_{\xi_C}$ of Proposition C.2.1, $\Delta_t^- \leq 0$ if $\dot{B}_t(V_{t+1}; x, \cdot)$ is convex for every $x \in X_t$ (the lower bound is certified), and $\Delta_t^+ \leq L_t^\xi e(\mathcal{P})$ if $\dot{B}_t(W; x, \cdot)$ is L_t^ξ -Lipschitz for $W \in \mathcal{W}_{t+1}$, as in Corollary 5.2(a). If the forward passes are drawn from $\hat{\mathbb{P}}$ (classical SDDP on the SAA), Theorem 3.5 controls the SAA gap; transferring it to the true cost needs regularity of the policy in ξ , which greedy argmin selections do not have in general.

C.4 Sampled cuts

At iteration k and for the cut of index $t < T$ (backward), draw $(\xi_{t,j}^k)_{j \in [N]}$ i.i.d. from $\mathbb{P}_{t+1}$ ($N = N_{k,t}$ deterministic), independently of everything drawn before. Here the state sets are convex with normalised Lebesgue measure (Lemma 2.3(a)), the cuts are affine, and $\underline{V}_t^0 \leq \mathcal{B}_t(\underline{V}_{t+1}^0)$.

Assumption F (Scenario oracle). For every $W \in \mathcal{W}_{t+1}^c$ (Appendix C.2), every $x \in X_t$ and every $\xi \in \Xi_{t+1}$, the oracle returns $a(\xi) = \dot{B}_t(W; x, \xi)$ and $g(\xi) \in E_t$ with $\|g(\xi)\| \leq L_t$ and $\dot{B}_t(W; y, \xi) \geq a(\xi) + \langle g(\xi), y - x \rangle$ for all $y \in X_t$, Borel in its inputs, and $\sup_\xi a(\xi) - \inf_\xi a(\xi) \leq \text{rg}_t$.

For instance, $\text{rg}_t \leq L_t^\xi \text{diam } \Xi_{t+1}$, or $\text{rg}_t \leq (T - t)\bar{\ell}$ if losses lie in $[0, \bar{\ell}]$ and $\underline{V}^0 = 0$. The raw sampled cut is $\hat{f}(y) := \bar{a} + \langle \bar{g}, y - x \rangle$, with $\bar{a}, \bar{g}$ the sample means of $a(\xi_{t,i}^k)$ and $g(\xi_{t,i}^k)$; it is valid in expectation only. Let $b(N, \delta) := (\text{rg}_t + 2L_t D_t) \sqrt{\ln(4/\delta)/N}$ (see Lemma C.4.2).

Theorem C.4.1 (Shifted sampled cuts). *Let [Assumptions A](#) and [F](#) hold, with a Euclidean norm on E_t in which $D_t \geq \text{diam } X_t$, and assume convex state sets, convex L_t -Lipschitz $\underline{V}_t^0$ with $\underline{V}_t^0 \leq \mathcal{B}_t(\underline{V}_{t+1}^0)$, and exact or γ^F -optimal forward passes from the true laws, with deterministic $\gamma_t^{F,k}$. Choose $\delta_{k,t} > 0$ with $\delta' := \sum_{k \geq 1, t < T} \delta_{k,t} \leq 1$. Let $b_{k,t} := b(N_{k,t}, \delta_{k,t})$, add the shifted cut $\hat{f} - b_{k,t}$ at iteration k and index t , and set $\Gamma_k := 2 \sum_{t < T} b_{k,t}$; let Γ_k^F be the forward-error sum (9). There is an event A with $\mathbb{P}(A) \geq 1 - \delta'$ on which every cut is valid ($\underline{V}_t^k \leq V_t$ for all k, t), and*

- (i) $\mathbb{E}[\mathbf{1}_A \sum_{k \geq 1} \Psi_T((G^k - \Gamma_k^F - \Gamma_k)_+)] \leq (1 + \delta'/2)M^V$, where $M^V := \sum_{t < T} \sup_{X_t}(V_t - \underline{V}_t^0)$;
- (ii) for $\delta \in (0, 1]$, with probability at least $1 - \delta - \delta'$, simultaneously for all $\theta > 0$, $\#\{k \geq 1 : G^k > \theta + \Gamma_k^F + \Gamma_k\} \leq (9 \ln(3/\delta) + \delta'/2)M^V / \Psi_T(\theta)$.

With $\delta_{k,t} = \delta'/(Tk(k+1))$, the condition $\Gamma_k \leq g_k$ for a target $g_k > 0$ holds as soon as $N_{k,t} \geq (2T(\text{rg}_t + 2L_t D_t)/g_k)^2 \ln(4Tk(k+1)/\delta')$.

Let $\mathcal{G}$ be the σ -field generated by everything drawn before the sample of iteration k and index t ; $x := \mathbf{x}_t^k$ and $W := \underline{V}_{t+1}^k$ are $\mathcal{G}$ -measurable. The conditional mean of the raw cut, $m(y) := \mathbb{E}[\hat{f}(y) \mid \mathcal{G}] = \mathcal{B}_t(W)(x) + \langle \mathbb{E}[g \mid \mathcal{G}], y - x \rangle$, satisfies $m \leq \mathcal{B}_t(W)$ on X_t .

Lemma C.4.2 (Uniform deviation of a sampled affine cut). *Let [Assumption F](#) hold with a Euclidean (Hilbert) norm on E_t , and let $\delta \in (0, 1]$. For every event $E \in \mathcal{G}$ on which $W \in \mathcal{W}_{t+1}^c$, almost surely on E , $\mathbb{P}(\sup_{y \in X_t} |\hat{f}(y) - m(y)| > b(N, \delta) \mid \mathcal{G}) \leq \delta$.*

Proof. Conditionally on $\mathcal{G}$ the pairs $(a, g)(\xi_{t,i}^k)$ are i.i.d. with the law obtained by freezing (x, W) . For $y \in X_t$, $\|y - x\| \leq D_t$, so $\sup_y |\hat{f}(y) - m(y)| \leq |\bar{a} - \mathbb{E}a| + D_t \|\bar{g} - \mathbb{E}g\|$. Hoeffding's inequality gives $\mathbb{P}(|\bar{a} - \mathbb{E}a| > \text{rg}_t \sqrt{\ln(4/\delta)/(2N)}) \leq \delta/2$. The map $(\xi_i)_i \mapsto \|\bar{g} - \mathbb{E}g\|$ has bounded differences $2L_t/N$, and $\mathbb{E}\|\bar{g} - \mathbb{E}g\| \leq L_t/\sqrt{N}$ by the Hilbert structure, so McDiarmid's inequality gives $\mathbb{P}(\|\bar{g} - \mathbb{E}g\| > L_t(1 + \sqrt{2 \ln(2/\delta)})/\sqrt{N}) \leq \delta/2$. Finally $1 + \sqrt{2v} \leq 2\sqrt{v + \ln 2}$ for $v \geq 0$, since $4v + 4 \ln 2 - (1 + \sqrt{2v})^2 = (\sqrt{2v} - 1)^2 + 4 \ln 2 - 2 > 0$; with $v = \ln(2/\delta)$ the sum of the two deviations is at most $b(N, \delta)$. $\square$

Proof of [Theorem C.4.1](#). Good events. Let $A_{k,t} := \{\sup_{X_t} |\hat{f} - m| \leq b_{k,t}\}$, Ω_0 the whole space and $\Omega_k := \Omega_{k-1} \cap \bigcap_{t < T} A_{k,t}$, which is $\mathcal{A}_k$ -measurable. On Ω_{k-1} , all cuts of iterations $\leq k-1$ are valid, so [Lemma 2.2\(a\)](#) gives $\underline{V}^j \leq V$ for $j \leq k-1$. On $E := \Omega_{k-1} \cap \bigcap_{s > t} A_{k,s} \in \mathcal{G}$, the cuts of indices $s > t$ of iteration k are valid too, so $W = \underline{V}_{t+1}^k$, the maximum of the convex $\underline{V}_{t+1}^0$ and of valid affine L_{t+1} -Lipschitz cuts, lies in $\mathcal{W}_{t+1}^c$. On $E \cap A_{k,t}$, with $b = b_{k,t}$, we have $\hat{f} - b \leq m \leq \mathcal{B}_t(W)$ on X_t (validity) and $(\hat{f} - b)(x) = \bar{a} - b \geq \mathcal{B}_t(W)(x) - 2b$. By [Lemma C.4.2](#), the probability that E holds while $A_{k,t}$ fails is at most $\delta_{k,t}$, also conditionally on $\mathcal{A}_{k-1}$. Summing over the first failing index, $\mathbb{P}(\Omega_{k-1} \setminus \Omega_k \mid \mathcal{A}_{k-1}) \leq \sum_{t < T} \delta_{k,t}$. Let $A := \bigcap_k \Omega_k$; then $\mathbb{P}(A^c) \leq \delta'$.

Lift on Ω_k . There, every cut so far is valid and L_t -Lipschitz, and the shifted cut at x is at least $\mathcal{B}_t(\underline{V}_{t+1}^{k-1})(x) - 2b_{k,t}$. As in the proof of [Theorem 3.5](#), $\Phi^k - \Phi^{k-1} \geq \Psi_T((R^k - \Gamma_k)_+)$. On Ω_{k-1} , $\underline{V}^{k-1} \leq V$, so $r_t^k \in [0, \sup(V_t - \underline{V}_t^0)]$ and $R^k \in [0, M^V]$. Also $\Psi_T(u) \leq \psi_0(u) \leq u/2$ by (13).

Stopped potential. Let $\tau_f := \inf\{k : \Omega_k \text{ fails}\}$, a stopping time, and $Y_k := \Phi^{k \wedge (\tau_f - 1)}$. The process (Y_k) is adapted, nondecreasing and bounded by M^V , and $Y_k - Y_{k-1} = \mathbf{1}_{\Omega_k}(\Phi^k - \Phi^{k-1})$. Using $\Omega_{k-1} \in \mathcal{A}_{k-1}$, conditional Jensen and [Lemma 3.2](#),

$$\mathbb{E}[Y_k - Y_{k-1} \mid \mathcal{A}_{k-1}] \geq \mathbf{1}_{\Omega_{k-1}} \Psi_T((G^k - \Gamma_k^F - \Gamma_k)_+) - \frac{M^V}{2} \mathbb{P}(\Omega_{k-1} \setminus \Omega_k \mid \mathcal{A}_{k-1}). \quad (51)$$

Summing and taking expectations gives (i), since $\mathbb{E} \lim_k Y_k \leq M^V$, $\sum_k \mathbb{P}(\Omega_{k-1} \setminus \Omega_k) \leq \delta'$ and $\mathbf{1}_A \leq \mathbf{1}_{\Omega_{k-1}}$. For (ii), apply [Lemma 3.4](#) to $M^V - Y_k$: with probability at least $1 - \delta$ its regret is at most $9M^V \ln(3/\delta)$, and by (51) it is at least $\sum_{k \geq 1} \mathbf{1}_{\Omega_{k-1}} \Psi_T((G^k - \Gamma_k^F - \Gamma_k)_+) - M^V \delta'/2$; intersect with A . Finally, $2 \sum_{t < T} b_{k,t} \leq g_k$ if each $b_{k,t} \leq g_k/(2T)$; solving for N gives the sample-size condition. $\square$

Proposition C.4.3 (Raw sampled cuts can stall SDDP). *Let $T = 2$, $X_0 = \{x_0\}$, $\mathcal{X}_1(x_0, \xi) = [0, 1]$, $\ell_1 \equiv 0$, $X_1 = [0, 1]$, $\mathcal{X}_2(x, \xi) = \{0\}$, $\ell_2(x, 0, \xi) = \xi x + 0.6(1 - x)$ with $\xi_2 \sim U[0, 1]$, and $\underline{V}^0 \equiv 0$. Then $V_1(x) = 0.6 - 0.1x$ and $V_0(x_0) = 0.5$, attained only at $x_1 = 1$. Run SDDP with exact forward passes and raw sampled cuts at index 1, built from fresh samples of sizes $N_k \geq 1$. If the cut of iteration j is built from a sample with mean $\bar{\xi}_j > 0.6$, then at every iteration $k > j$ the policy plays $x_1 = 0$, $V_0^{\pi^k}(x_0) = 0.6 = V_0(x_0) + 0.1$ and $\underline{V}_0^{k-1}(x_0) = 0.6 > V_0(x_0)$; in particular $G^k = 0$ although π^k is 0.1-suboptimal. This happens with probability at least $\mathbb{P}(\bar{\xi}_1 > 0.6) > 0$ for every choice of sample sizes, and almost surely if $\sup_k N_k < \infty$.*

Proof of Proposition C.4.3. For every ξ , $y \mapsto \bar{\mathcal{B}}_1(0; y, \xi) = \xi y + 0.6(1 - y)$ is affine, so its exact scenario cut is itself, and a raw sampled cut is the line $f_j(y) = \bar{\xi}_j y + 0.6(1 - y)$, through $(0, 0.6)$ and $(1, \bar{\xi}_j)$. Hence $\underline{V}_1^k(0) = 0.6$ for $k \geq 1$. If $\bar{\xi}_j > 0.6$ for some $j \leq k - 1$, then $\underline{V}_1^{k-1}(y) \geq 0.6 + (\bar{\xi}_j - 0.6)y > 0.6$ for $y \in (0, 1]$, so the forward pass of iteration k plays $x_1 = 0$. Index 0 is deterministic with $d_0 = 0$; its exact cut gives $\underline{V}_0^{k-1}(x_0) = \min_{[0,1]} \underline{V}_1^{k-1} = 0.6$, while $V_0^{\pi^k}(x_0) = V_1(0) = 0.6$. Cuts are never removed, so this persists. The mean of N i.i.d. $U[0, 1]$ variables has a positive density on $(0, 1)$, so $\mathbb{P}(\bar{\xi} > 0.6) > 0$. The cuts use independent samples, so if $N_k \leq \bar{N}$ for all k , an exceedance occurs almost surely. $\square$

C.5 Noise with unbounded support

With convex V_t and an unbounded reachable set, costs and residuals are typically unbounded (a convex function bounded on $\mathbb{R}^n$ is constant); a tail term replaces the uniform bound.

Proposition C.5.1 (Leaving the compact set). *Let $X'_t \supset X_t$ be Borel sets with $\emptyset \neq \mathcal{X}_t(x, \xi) \subset X'_t$ for $x \in X'_{t-1}$, $\xi \in \Xi_t$, where the X_t are convex and compact with normalised Lebesgue measure. Let Assumption A hold with X'_t in place of X_t , except that compactness is dropped and the uniform bound in (S2) is replaced by integrability of ℓ_t , V_t and $\underline{V}_t^0$ along the forward paths. Assume that $\underline{V}_t^0 \leq V_t$ on X'_t and that $\underline{V}_t^0$ is L_t -Lipschitz on X_t , that the cuts are valid on X'_t (below V_t), L_t -Lipschitz on X_t and γ -tight as in Section 5.2, and that for all k*

$$\Delta_k^{\text{out}} := \mathbb{E} \left[\sum_{t < T} r_t^k(\mathbf{x}_t^k) \mathbf{1}\{\mathbf{x}_t^k \notin X_t\} \mid \mathcal{A}_{k-1} \right] \leq \Delta^{\text{out}}. \quad (52)$$

Then $\mathbb{E}[\sum_{k \geq 1} \Psi_T((G^k - \bar{\Gamma}_k - \Delta^{\text{out}})_+)] \leq M^V := \sum_{t < T} \sup_{X_t} (V_t - \underline{V}_t^0)$. The bound on Δ_k^{out} holds in either of two cases: $r_t^k \leq M'_t$ on X'_t for some $M'_t \geq 0$ and $\mathbb{P}(\mathbf{x}_t^k \notin X_t \mid \mathcal{A}_{k-1}) \leq o_t$, with $\Delta^{\text{out}} = \sum_t M'_t o_t$; or $\mathbb{E}[(V_t - \underline{V}_t^0)(\mathbf{x}_t^k)^2 \mid \mathcal{A}_{k-1}] \leq \bar{m}_t^2$ and the same o_t , with $\Delta^{\text{out}} = \sum_t \bar{m}_t \sqrt{o_t}$.

Proof. Validity on X'_t and $\underline{V}_t^0 \leq V_t$ give $\underline{V}_t^0 \leq \underline{V}_t^k \leq V_t$ on X'_t . Since $\mathcal{X}_{t+1}(x, \xi) \subset X'_{t+1}$ for $x \in X'_t$ and $\mathcal{B}_t$ is monotone, $r_t^k \leq \mathcal{B}_t(V_{t+1}) - \underline{V}_t^0 = V_t - \underline{V}_t^0$ there (residuals may be negative). Lemma 3.2 holds with X'_t in place of X_t , by the same telescoping: $\underline{V}_t^{k-1}(\mathbf{x}_t^k)$ is integrable because $\underline{V}_t^0 \leq \underline{V}_t^{k-1} \leq V_t$, and so is every $r_t^k(\mathbf{x}_t^k)$, by the one-step identity (11). The potential is computed on the compact sets X_t , where it is at most M^V , and Lemma 3.3 is applied only at trial points in X_t . Let $\tilde{R}^k := \sum_{t < T} r_t^k(\mathbf{x}_t^k) \mathbf{1}\{\mathbf{x}_t^k \in X_t\}$, which is integrable and at most M^V . Since $\sum_{t < T} (r_t^k(\mathbf{x}_t^k) - \gamma_t^k) \mathbf{1}\{\mathbf{x}_t^k \in X_t\} \geq (\tilde{R}^k - \Gamma_k)_+$, we get $\Phi^k - \Phi^{k-1} \geq \Psi_T((\tilde{R}^k - \Gamma_k)_+)$. Then $\mathbb{E}[\tilde{R}^k \mid \mathcal{A}_{k-1}] = \mathbb{E}[R^k \mid \mathcal{A}_{k-1}] - \Delta_k^{\text{out}} \geq G^k - \Gamma_k^F - \Delta_k^{\text{out}}$, conclude as in Theorem 3.5. For the second sufficient condition, use Cauchy-Schwarz and $r_t^k \leq V_t - \underline{V}_t^0$. $\square$

D Proofs and complements for Section 6

D.1 Standard facts, measurability and the four lemmas

Let $\mathfrak{M}$ be the set of bounded Borel $W : X \rightarrow \mathbb{R}$ such that $(x, \xi) \mapsto \dot{\mathcal{B}}(W; x, \xi)$ is jointly measurable; by (S3_∞), it contains the models of the algorithm and V , and $\mathcal{B}$ is defined on it.

Lemma D.1.1 (Standard facts). *Under Assumption C: (i) for $W, W' \in \mathfrak{M}$, $\mathcal{B}W$ is Borel and bounded, $W \leq W'$ implies $\mathcal{B}W \leq \mathcal{B}W'$, $\mathcal{B}(W + c) = \mathcal{B}W + \beta c$ for constants c , so $\sup |\mathcal{B}W - \mathcal{B}W'| \leq \beta \sup |W - W'|$; and $\mathfrak{M}$ is closed under uniform convergence; (ii) $0 \leq V \leq \bar{\ell}/(1 - \beta)$, V is the only bounded fixed point of $\mathcal{B}$ in $\mathfrak{M}$, every $W \in \mathfrak{M}$ with $W \leq \mathcal{B}W$ satisfies $W \leq V$, and $\mathcal{B}^j W \rightarrow V$ uniformly whenever $\mathcal{B}^j W \in \mathfrak{M}$ for all $j \geq 0$; (iii) $V^\pi(x_0) \geq \mathbb{E}[\ell(x_0, \mathbf{x}_1, \boldsymbol{\xi}_1) + \beta V(\mathbf{x}_1)] \geq V(x_0)$ for every policy π , also conditionally on a σ -field independent of the noise on which π may depend (as π^k depends on $\mathcal{A}_{k-1}$).*

Proof. (i) Since $\emptyset \neq \mathcal{X}(x, \xi) \subset X$ and $\ell \in [0, \bar{\ell}]$, $\beta \inf W \leq \dot{\mathcal{B}}(W; x, \xi) \leq \bar{\ell} + \beta \sup W$, and Fubini's theorem makes $\mathcal{B}W$ Borel; monotonicity and the shift are immediate, and give the contraction. If $W_j \rightarrow W$ uniformly, $|\dot{\mathcal{B}}(W_j; x, \xi) - \dot{\mathcal{B}}(W; x, \xi)| \leq \beta \sup |W_j - W|$, so $\dot{\mathcal{B}}(W; \cdot, \cdot)$ is a pointwise limit of jointly measurable functions. (ii) $V = \mathcal{B}V \geq \beta \inf V$ gives $\inf V \geq 0$, and $V \leq \bar{\ell} + \beta \sup V$ gives $\sup V \leq \bar{\ell}/(1 - \beta)$. If $W \leq \mathcal{B}W$ and $c := \sup(W - V)_+$, then $W \leq V + c$ and $W \leq \mathcal{B}W \leq \mathcal{B}(V + c) = V + \beta c$, so $c \leq \beta c$ and $c = 0$. Uniqueness and convergence follow from the contraction. (iii) Let $\lambda_t := \ell(\mathbf{x}_t, \mathbf{x}_{t+1}, \boldsymbol{\xi}_{t+1})$ and $\mathcal{H}_t$ the σ -field of the conditioning and of $\boldsymbol{\xi}_{[t]}$. As $\lambda_t + \beta V(\mathbf{x}_{t+1}) \geq \dot{\mathcal{B}}(V; \mathbf{x}_t, \boldsymbol{\xi}_{t+1})$, the freezing property gives $\mathbb{E}[\lambda_t + \beta V(\mathbf{x}_{t+1}) | \mathcal{H}_t] \geq V(\mathbf{x}_t)$. Iterate from $t = 1$ and let the horizon tend to infinity (V is bounded): $V^\pi(x_0) \geq \mathbb{E}[\lambda_0 + \beta V(\mathbf{x}_1)]$, and $t = 0$ gives $\geq V(x_0)$. $\square$

A sufficient condition for (S3_∞). Assume the sufficient condition of Appendix A.4 and, moreover, that $(x, y) \mapsto \ell(x, y, \xi) + \delta_{\mathcal{X}(x, \xi)}(y)$ is lower semicontinuous on $X \times X$ for every ξ . For a bounded lower semicontinuous W , $\dot{\mathcal{B}}(W; \cdot, \cdot)$ is then jointly measurable (Appendix A.4), and $\dot{\mathcal{B}}(W; \cdot, \xi)$, an infimum over the compact X , is lower semicontinuous and at least $\beta \inf W$, so $\mathcal{B}W$ is lower semicontinuous (Fatou). Hence the $\mathcal{B}^j 0$ lie in $\mathfrak{M}$, and their uniform limit V is in $\mathfrak{M}$ and satisfies $\mathcal{B}V = V$ (Lemma D.1.1(i)–(ii)). For finitely supported noise and deterministic oracles, (S3_∞) is not needed: $\mathcal{B}$ is defined on all bounded functions, and all the random variables take countably many values.

Lemma D.1.2 (Sufficient conditions). *Assume Assumption C, and let every cut be computed for the current model W . (a) If $\underline{V}^0 \leq V$ and every cut satisfies $f \leq \mathcal{B}W$ on X , then every model is $\leq V$, so (O3_∞) holds with $U = V$ and $M_U \leq \bar{\ell}/(1 - \beta) - \inf_X \underline{V}^0$. If moreover $\underline{V}^0 \leq \mathcal{B}\underline{V}^0$ (e.g. $\underline{V}^0 = 0$), the assumption $\underline{V}^0 \leq V$ holds by Lemma D.1.1(ii), and every model satisfies $W \leq \mathcal{B}W$, so $r^k \geq 0$. (b) Under (O4_∞), if $f(x) \geq \mathcal{B}W(x) - \gamma_{k,t}$ at $x = \mathbf{x}_t^k$, then (O2_∞) holds.*

Proof. (a) If $\underline{V}^0 \leq \mathcal{B}\underline{V}^0$, then $\underline{V}^0 \leq V$ by Lemma D.1.1(ii). By induction on the cuts, let $W \leq V$ be the current model and f a cut; then $f \leq \mathcal{B}W \leq \mathcal{B}V = V$, so $\max(W, f) \leq V$. If moreover $W \leq \mathcal{B}W$, then, with $W' := \max(W, f) \geq W$, $W \leq \mathcal{B}W \leq \mathcal{B}W'$ and $f \leq \mathcal{B}W \leq \mathcal{B}W'$, so $W' \leq \mathcal{B}W'$. (b) By (O4_∞), $W \geq \underline{V}^{k-1}$, and $\mathcal{B}$ is monotone. $\square$

Proof of Lemma 6.1. Write $\underline{V} := \underline{V}^{k-1}$, $r_t := r^k(\mathbf{x}_t)$ and $\mathcal{H}_t := \mathcal{A}_{k-1} \vee \sigma(\boldsymbol{\xi}_{[t]})$. The bounds on e_t are the definitions of π^k and $\dot{\mathcal{B}}$. By the freezing property and (S3_∞), $\mathbb{E}[\dot{\mathcal{B}}(\underline{V}; \mathbf{x}_t, \boldsymbol{\xi}_{t+1}) | \mathcal{H}_t] = \mathcal{B}\underline{V}(\mathbf{x}_t)$, so $\mathbb{E}[\lambda_t + \beta \underline{V}(\mathbf{x}_{t+1}) | \mathcal{H}_t] = \underline{V}(\mathbf{x}_t) + r_t + \mathbb{E}[e_t | \mathcal{H}_t]$. Multiply by β^t , take $\mathbb{E}_{k-1}$, sum over $H \leq t < H'$ and let $H' \rightarrow \infty$; all terms are bounded given $\mathcal{A}_{k-1}$, and $\beta^{H'} \mathbb{E}_{k-1} \underline{V}(\mathbf{x}_{H'}) \rightarrow 0$, so

$\varepsilon_H^k = \mathbb{E}_{k-1} \sum_{t \geq H} \beta^t (r_t + e_t)$. For $H = 0$ this is G^k , as $\mathbf{x}_0 = x_0$; subtracting gives (34), and $\lambda_t \leq \bar{\ell}$ bounds ε_H^k . For τ , Fubini and independence give $\mathbb{E}_{k-1} \sum_{t < \tau} (r_t + e_t) = \sum_t \beta^t \mathbb{E}_{k-1} (r_t + e_t)$. $\square$

Proof of Lemma 6.2. The bounds stated after Assumption D hold as in the proof of Lemma 3.1(i) (an $\hat{L}$ -Lipschitz $f \leq U$ satisfies $f \leq \bar{U}$, and $\bar{U} \leq M_U + \sup V^0$), and give those on Φ^k . Fix $t < \tau_k$, $x := \mathbf{x}_t^k$ and the cut f at x . By (O2 $_\infty$) and (O4 $_\infty$), $\underline{V}^k(x) \geq f(x) \geq \mathcal{B}\underline{V}^{k-1}(x) - \gamma_{k,t} = \underline{V}^{k-1}(x) + r_t^k(x) - \gamma_{k,t}$ and $\underline{V}^k \geq \underline{V}^{k-1}$, so $\underline{V}^k(x) \geq \underline{V}^{k-1}(x) + s_t^k$. Lemma 3.3, applied to $V = \underline{V}^{k-1}$ and $W = \underline{V}^k$, gives $\Phi^k - \Phi^{k-1} \geq \kappa \hat{\varphi}(s_t^k)$. Take the maximum over t ; $\hat{\varphi}$ is nondecreasing and $\hat{\psi} \leq \hat{\varphi}$. $\square$

Lemma D.1.3 (Pass lengths). *Let $(r_t)_{t \geq 0} \subset [0, R]$, $\bar{r}_0 := 0$, $\bar{r}_n := \max_{t < n} r_t$ ($n \geq 1$), and τ a random variable in $\mathbb{N}^*$.*

- (i) $\mathbb{E} \bar{r}_\tau = \sum_{s \geq 0} \mathbb{P}(\tau > s) (\bar{r}_{s+1} - \bar{r}_s)$.
- (ii) *If $\mathbb{P}(\tau > s) \geq \beta^s$ for all $s \geq 0$, then $\mathbb{E} \bar{r}_\tau \geq (1 - \beta) \sum_{n \geq 1} \beta^{n-1} \bar{r}_n \geq (1 - \beta) \sum_{t \geq 0} \beta^t r_t$. Conversely, if $R > 0$ and $\mathbb{E} \bar{r}_\tau \geq (1 - \beta) \sum_t \beta^t r_t$ for every such sequence, then $\mathbb{P}(\tau > s) \geq \beta^s$ for all s . Under this condition $\mathbb{E} \tau \geq 1/(1 - \beta)$, with equality iff $\mathbb{P}(\tau > s) = \beta^s$ for all s .*
- (iii) *For a deterministic $\tau = H$: $\bar{r}_H \geq \frac{1-\beta}{1-\beta^H} \sum_{t < H} \beta^t r_t \geq (1 - \beta) \sum_{t < H} \beta^t r_t$.*

Proof. (i) Write $\bar{r}_n = \sum_{s < n} (\bar{r}_{s+1} - \bar{r}_s)$ and use Tonelli; the increments are nonnegative. (ii) Hence $\mathbb{E} \bar{r}_\tau \geq \sum_s \beta^s (\bar{r}_{s+1} - \bar{r}_s) = (1 - \beta) \sum_{n \geq 1} \beta^{n-1} \bar{r}_n$ (the series converge since $\bar{r}$ is bounded); then use $\bar{r}_n \geq r_{n-1}$. For the converse, take $r_t = R \mathbf{1}\{t \geq s\}$: then $\bar{r}_n = R \mathbf{1}\{n > s\}$, $\mathbb{E} \bar{r}_\tau = R \mathbb{P}(\tau > s)$ and $(1 - \beta) \sum_{t \geq s} \beta^t R = R \beta^s$. Finally $\mathbb{E} \tau = \sum_s \mathbb{P}(\tau > s) \geq \sum_s \beta^s$. (iii) $\sum_{t < H} \beta^t r_t \leq \bar{r}_H (1 - \beta^H)/(1 - \beta)$. $\square$

D.2 Periodic problems and proof of the master theorem

Periodic model. Lemma D.1.1 and its proof hold for the periodic model, with $\mathbf{T} : (W_p)_{p \in \mathbb{Z}_S} \mapsto (\mathcal{B}_p W_{p+1})_{p \in \mathbb{Z}_S}$ in place of $\mathcal{B}$, the norm $\max_p \sup_{X_p} |W_p|$, the classes $\mathfrak{M}_p$, and, in (iii), the cost from phase 0 and $\mathcal{B}_{p(t)}$ at time t . In the proof of Lemma D.1.2(a), adding a cut $f \leq \mathcal{B}_p W_{p+1}$ to W_p gives a family W' with $W'_{p'} \leq \mathcal{B}_{p'} W_{p'+1} \leq \mathcal{B}_{p'} W'_{p'+1}$ for all p' , and $f \leq \mathcal{B}_p W_{p+1} = V_p$. Lemmas 6.1 and 6.2 hold with $r_t := r_{p(t)}^k(\mathbf{x}_t)$ and $\underline{V}_{p(t)}^{k-1}$ at time t . We identify $\mathbb{Z}_S$ with $\{p \in \mathbb{N} : p < S\}$. The number of visits to a phase before the end of a geometric pass is not geometric, but the summation by parts of Lemma D.1.3 still applies.

Lemma D.2.1 (Pass lengths, phase by phase). *Let $(r_t)_{t \geq 0} \subset [0, R]$, let $\tau \in \mathbb{N}^*$ satisfy $\mathbb{P}(\tau > s) = \beta^s$ for all $s \geq 0$, and, for $p \in \mathbb{Z}_S$, let $\bar{r}_{(p)} := \max\{r_t : t < \tau, p(t) = p\}$, with $\max \emptyset := 0$. Then $\mathbb{E} \bar{r}_{(p)} \geq (1 - \beta^S) \sum_{t \geq 0: p(t)=p} \beta^t r_t$.*

Proof. Let $q_j := r_{p+jS}$ for $j \geq 0$, $\bar{q}_0 := 0$, $\bar{q}_n := \max_{j < n} q_j$ for $n \geq 1$, and $N := \#\{j \geq 0 : p + jS < \tau\}$, so that $\bar{r}_{(p)} = \bar{q}_N$ and $\mathbb{P}(N > j) = \beta^{p+jS}$. The identity of Lemma D.1.3(i) holds for every random variable with values in $\mathbb{N}$, so $\mathbb{E} \bar{r}_{(p)} = \sum_{j \geq 0} \beta^{p+jS} (\bar{q}_{j+1} - \bar{q}_j) = \beta^p (1 - \beta^S) \sum_{n \geq 1} \beta^{S(n-1)} \bar{q}_n \geq \beta^p (1 - \beta^S) \sum_{j \geq 0} \beta^{jS} q_j$, since $\bar{q}$ is bounded and $\bar{q}_n \geq q_{n-1}$. $\square$

Let

$$\chi_S(g) := \max \left\{ \hat{\psi}((1 - \beta)g), S \hat{\psi} \left(\frac{(1 - \beta^S)g}{S} \right) \right\} \quad (g \geq 0). \quad (53)$$

The first term credits a pass with its largest residual, as in Theorem 6.3, against a potential S times larger; alone, it gives Theorem D.4.1 with the periodic $\bar{M}_U$ for deterministic pass lengths. The second credits a pass with one residual per phase, but a pass visits a given phase only about $1/(S(1 - \beta))$ times, so the factor $1 - \beta$ becomes $(1 - \beta^S)/S = (1 - \beta)/q_S$, with q_S as in Theorem 6.5. Since $1 - \beta^S = (1 - \beta) \sum_{i < S} \beta^i \leq S(1 - \beta)$, $q_S \geq 1$; since $1 - \beta^S \geq 1 - \beta$, $q_S \leq S$; with $x := S(1 - \beta)$, $\ln \beta \leq \beta - 1$ gives $\beta^S \leq e^{-x}$, so $q_S \leq x/(1 - e^{-x}) \leq 1 + x$, because $(1 + x)e^{-x} \leq 1$.

Proof of Theorem 6.5, hence of Theorem 6.3(a). Fix $k \geq 1$, couple the pass with an infinite trajectory $(\mathbf{x}_t)_{t \geq 0}$ of π^k from x_0 , independent of τ_k given $\mathcal{A}_{k-1}$, and let $r_t := r_{p(t)}^k(\mathbf{x}_t)$ and $s_t := (r_t - \gamma_{k,t})_+$. On the $\mathcal{A}_{k-1}$ -measurable event where $\sum_t \beta^t \gamma_{k,t} = \infty$, $\Gamma_k^\beta = \infty$ and there is nothing to prove, since $\Phi^k \geq \Phi^{k-1}$; we work on its complement. Models, residuals and s_t are bounded (Lemma 6.2).

Gap identity. (i) is Lemma 6.1 with $\tau = \tau_k$, with (33), which follows from Lemma D.1.1(iii) and $V_0^{k-1}(x_0) \leq \bar{U}_0(x_0) \leq U_0(x_0)$. By Lemma 6.1 with $H = 0$ and $e_t \leq \gamma_k^F$,

$$\mathbb{E}_{k-1} \sum_{t \geq 0} \beta^t (r_t - \gamma_{k,t}) \geq G^k - \Gamma_k^\beta. \quad (54)$$

Pathwise lift. As in the proof of Lemma 6.2, at $t < \tau_k$, with $x := \mathbf{x}_t^k$ and $p := p(t)$, $V_p^k(x) \geq V_p^{k-1}(x) + s_t$ and $V_p^k \geq V_p^{k-1}$, so $\int (V_p^k - V_p^{k-1}) d\mu_p \geq \kappa \hat{\psi}(\bar{s}_p)$ with $\bar{s}_p := \max\{s_t : t < \tau_k, p(t) = p\}$ ($\max \emptyset := 0$). Summing over p ,

$$\Phi^k - \Phi^{k-1} \geq \Lambda_k := \kappa \sum_{p \in \mathbb{Z}_S} \hat{\psi}(\bar{s}_p), \quad (55)$$

which is (38), and implies Lemma 6.2 ($S = 1$).

Lower bounds. First, $\Lambda_k \geq \kappa \hat{\psi}(\bar{s})$ with $\bar{s} := \max_{t < \tau_k} s_t = \max_p \bar{s}_p$. Given $\mathcal{A}_{k-1}$ and the path, Lemma D.1.3(ii) gives $\mathbb{E}[\bar{s} \mid \mathcal{A}_{k-1}, \text{path}] \geq (1 - \beta) \sum_t \beta^t s_t$, so $\mathbb{E}_{k-1} \bar{s} \geq (1 - \beta)(G^k - \Gamma_k^\beta)_+$ by $s_t \geq \max\{0, r_t - \gamma_{k,t}\}$ and (54), and conditional Jensen ($\hat{\psi}$ is convex and $\bar{s}$ is bounded) gives $\mathbb{E}_{k-1} \Lambda_k \geq \kappa \hat{\psi}((1 - \beta)(G^k - \Gamma_k^\beta)_+)$. Second, by Lemma D.2.1, $a_p := \mathbb{E}_{k-1} \bar{s}_p \geq 0$ satisfies $\sum_p a_p \geq (1 - \beta^S) \mathbb{E}_{k-1} \sum_t \beta^t s_t \geq (1 - \beta^S)(G^k - \Gamma_k^\beta)_+$, and conditional Jensen, convexity and monotonicity of $\hat{\psi}$ give $\mathbb{E}_{k-1} \Lambda_k \geq \kappa \sum_p \hat{\psi}(a_p) \geq \kappa S \hat{\psi}(\sum_p a_p / S) \geq \kappa S \hat{\psi}((1 - \beta^S)(G^k - \Gamma_k^\beta)_+ / S)$. Hence $\mathbb{E}_{k-1}[\Phi^k - \Phi^{k-1}] \geq \mathbb{E}_{k-1} \Lambda_k \geq \kappa \chi_S((G^k - \Gamma_k^\beta)_+)$.

Summation and rates. Sum over k , with $0 = \Phi^0 \leq \Phi^k \leq \bar{M}_U$. On $\{G^k > \theta + \Gamma_k^\beta\}$ the k -th term is at least $\kappa \chi_S(\theta)$, as χ_S is nondecreasing. If $(1 - \beta)\theta \leq \hat{\rho}$, then $(1 - \beta^S)\theta / S = (1 - \beta)\theta / q_S \leq \hat{\rho}$ too, both evaluations of $\hat{\psi}$ are on its first branch, and $\chi_S(\theta) = \max\{1, S/q_S^{d+1}\}((1 - \beta)\theta)^{d+1} / (2(d+1)\hat{\rho}^d)$. For (iii), apply Lemma 3.4 with $Y_k := \bar{M}_U - \Phi^k$ ($Y_0 = \bar{M}_U$ is deterministic) and $f_k := \kappa \chi_S((G^k - \Gamma_k^\beta)_+)$, which is $\mathcal{A}_{k-1}$ -measurable. For $S = 1$, $q_S = 1$ and these are the statements of Theorem 6.3(a); the cut count follows from $\mathbb{E}[\tau_k \mid \mathcal{A}_{k-1}] = 1/(1 - \beta)$. For SDDP, $\bar{M}_U \leq S\bar{\ell}/(1 - \beta)$ and $q_S \leq \min\{S, 1 + S(1 - \beta)\}$ give the bound stated after Theorem 6.5. $\square$

D.3 Covering versions

Proof of Theorem 6.3(b) and (c). Let $Y_k := \sum_{c \in C} (\bar{U} - V^k)(c)$; by the bounds after Assumption D, (Y_k) is adapted, nonincreasing and nonnegative, and $Y_0 = M_C$ is deterministic. Fix $k \geq 1$ and $t < \tau_k$, let $x := \mathbf{x}_t^k$, and choose $c \in C$ with $\text{dist}(x, c) \leq \sigma$. As in the proof of Lemma 6.2, $(V^k - V^{k-1})(x) \geq s_t^k$; both models are $\hat{L}$ -Lipschitz, so $(V^k - V^{k-1})(c) \geq (s_t^k - 2\hat{L}\sigma)_+$, and the other lifts are nonnegative. Different t may give the same c , so only the maximum is available: $Y_{k-1} - Y_k \geq (\bar{s}^k - 2\hat{L}\sigma)_+$. By conditional Jensen for $(\cdot)_+$, then $\mathbb{E}_{k-1} \bar{s}^k \geq (1 - \beta)(G^k - \Gamma_k^\beta)_+$ (proof of Theorem 6.5) and $2\hat{L}\sigma \leq (1 - \beta)\theta/2$,

$$\mathbb{E}_{k-1}[Y_{k-1} - Y_k] \geq (\mathbb{E}_{k-1} \bar{s}^k - 2\hat{L}\sigma)_+ \geq (1 - \beta)(G^k - \Gamma_k^\beta - \theta/2)_+ =: f_k.$$

Sum over $k \in [K]$ and let $K \rightarrow \infty$: $\mathbb{E} \sum_k f_k \leq Y_0 = M_C$. The count follows from $(\theta/2) \#\{k \geq 1 : G^k > \theta + \Gamma_k^\beta\} \leq \sum_{k \geq 1} (G^k - \Gamma_k^\beta - \theta/2)_+$, and the high-probability claim is Lemma 3.4 applied to (Y_k) , with $C = M_C$ and these f_k . The net size is Lemma A.1.1(a) with $2D/\sigma = 8\hat{L}D/((1 - \beta)\theta)$.

For (c), $\bar{U} = U$ because $\text{dist} = +\infty$ off the diagonal; take $C = X$, $c = x$, $\sigma = 0$ and $\theta = 0$, so that the Lipschitz property is not used, with $f_k = (1 - \beta)(G^k - \Gamma_k^\beta)_+$ and $Y_0 = C_{\text{fin}}$. For periodic problems, take one net C_p per phase, $Y_k := \sum_p \sum_{c \in C_p} (\bar{U}_p - V_p^k)(c)$ and $c \in C_{p(t)}$; the argument is unchanged. $\square$

D.4 Deterministic pass length

Theorem D.4.1 (Deterministic pass length). *Let Assumptions C and D and (VG_t) hold as in Lemma 6.2, with $\underline{V}^0 \geq 0$, and let $\hat{\rho}$, $\hat{\psi}$, Γ_k^β and $\bar{M}_U$ be as in Sections 6.2 and 6.3. Run variant (D) with pass length H ; let $\varepsilon_H := \beta^H \bar{\ell} / (1 - \beta)$ and $\varsigma \in (0, 1)$.*

- (i) $\mathbb{E}[\Phi^k - \Phi^{k-1} \mid \mathcal{A}_{k-1}] \geq \kappa \hat{\psi}((1 - \beta)(G^k - \Gamma_k^\beta - \varepsilon_H)_+)$ for $k \geq 1$, so the sum over k of the expectations of the right-hand sides is at most $\bar{M}_U$. If $\Gamma_k^\beta \leq \Gamma^\beta$ (deterministic), then, for $\theta > 0$ with $\varepsilon_H \leq \varsigma \theta$ and $(1 - \beta)\theta \leq \hat{\rho}$,

$$\mathbb{E} \# \{k \geq 1 : G^k > \theta + \Gamma^\beta\} \leq \frac{2(d+1) \bar{M}_U \hat{\rho}^d}{\kappa(1 - \varsigma)^{d+1}(1 - \beta)^{d+1}\theta^{d+1}}; \quad (56)$$

for $\delta \in (0, 1]$, with probability at least $1 - \delta$, simultaneously for all θ with $\varepsilon_H \leq \varsigma \theta$, $\# \{k \geq 1 : G^k > \theta + \Gamma_k^\beta\} \leq 9 \ln(3/\delta) \bar{M}_U / (\kappa \hat{\psi}((1 - \varsigma)(1 - \beta)\theta))$.

- (ii) For $\Gamma_k^\beta \leq \Gamma^\beta$, $0 < \theta < \bar{\ell} / (1 - \beta)$ with $(1 - \beta)\theta \leq \hat{\rho}$ and $H := \lceil \ln((d+2)\bar{\ell} / ((1 - \beta)\theta)) / (1 - \beta) \rceil$,

$$\mathbb{E} \# \{k \geq 1 : G^k > \theta + \Gamma^\beta\} \leq \frac{2e(d+1) \bar{M}_U \hat{\rho}^d}{\kappa(1 - \beta)^{d+1}\theta^{d+1}}, \quad (57)$$

which is at most $2e(d+1)\bar{\ell}(4\hat{L}D)^d / (\kappa(1 - \beta)^{d+2}\theta^{d+1})$ if $\bar{M}_U \leq \bar{\ell} / (1 - \beta)$, and the expected number of cuts (trial-point cuts; the optional cuts of (O4_∞) are not counted) before the first iteration with $G^k \leq \theta + \Gamma^\beta$ is at most H times (57).

Proof of Theorem D.4.1. (i) Let $s_t := s_t^k$ and $r_t := r_t^k(x_t^k)$. By Lemma 6.2 and Lemma D.1.3(iii), $\Phi^k - \Phi^{k-1} \geq \kappa \hat{\psi}((1 - \beta) \sum_{t < H} \beta^t s_t)$. By Lemma 6.1, $\mathbb{E}_{k-1} \sum_{t < H} \beta^t (r_t - \gamma_{k,t}) \geq G^k - \varepsilon_H^k - \Gamma_k^\beta$, and $\varepsilon_H^k \leq \varepsilon_H$ as $\underline{V}^{k-1} \geq \underline{V}^0 \geq 0$; since $s_t \geq \max\{0, r_t - \gamma_{k,t}\}$, $\mathbb{E}_{k-1} \sum_{t < H} \beta^t s_t \geq (G^k - \Gamma_k^\beta - \varepsilon_H)_+$. Conditional Jensen gives the first claim; sum over k , with $0 = \Phi^0 \leq \Phi^k \leq \bar{M}_U$. On $\{G^k > \theta + \Gamma^\beta\}$, $(G^k - \Gamma^\beta - \varepsilon_H)_+ \geq (1 - \varsigma)\theta$, so the k -th term is at least $\kappa \hat{\psi}((1 - \varsigma)(1 - \beta)\theta) = \kappa((1 - \varsigma)(1 - \beta)\theta)^{d+1} / (2(d+1)\hat{\rho}^d)$ (first branch). For the high-probability bound, apply Lemma 3.4 with $Y_k := \bar{M}_U - \Phi^k$; its regret is at least $\kappa \hat{\psi}((1 - \varsigma)(1 - \beta)\theta) \# \{k \in [K] : G^k > \theta + \Gamma_k^\beta\}$, and the exceptional event does not depend on θ . (ii) $\beta^H \leq e^{-(1-\beta)H} \leq (1 - \beta)\theta / ((d+2)\bar{\ell})$, so $\varepsilon_H \leq \theta / (d+2)$; take $\varsigma = 1/(d+2)$, which costs only $\ln(d+2)/(1 - \beta)$ more steps per pass than $\varsigma = 1/2$, and use $(1 + 1/(d+1))^{d+1} \leq e$. The first index with $G^k \leq \theta + \Gamma^\beta$ is at most $1 + \# \{k \geq 1 : G^k > \theta + \Gamma^\beta\}$. $\square$

D.5 Instances

Table 4 summarises the constants with which the oracles of this subsection satisfy Assumption D.

Proof of Corollary 6.6. Under (I1), $\mathcal{B}$ maps convex functions in $\mathfrak{M}$ to convex functions on X (on X_η under (E_η)): the partial infimum in y of the jointly convex $\ell(z, y, \xi) + \beta W(y)$ is convex in z , and finite; integration keeps convexity. So the models are convex, and subgradient cuts of $\mathcal{B}W$ satisfy (I3) where they exist. (O1_∞) is (I4), with $\underline{V}^0$ continuous, hence bounded. By (I2), (I3) and Lemma D.1.2, (O2_∞) holds with $\gamma_{k,t} = 0$, (O3_∞) with $U = V$, and $r^k \geq 0$; (O4_∞) holds by construction. Then $\zeta = 0$ and $\bar{U} \leq V$, so $\bar{M}_U \leq \bar{M} \leq \bar{\ell} / (1 - \beta)$, as $V \leq \bar{\ell} / (1 - \beta)$ and $\underline{V}^0 \geq 0$,

oracle	$\hat{L}$	γ	U	floor Γ^β	part
valid, γ -tight; Jensen	$\hat{L}$	$\gamma; L^\xi e(\mathcal{P})$	V	$\gamma/(1-\beta)$	(a)
SAA (Proposition D.5.1)	$\hat{L}$	$(\Delta^+)_+$	$\hat{V}$	$(\Delta^+)_+/(1-\beta);$ $\zeta \leq (\Delta^-)_+/(1-\beta)$	(a)
SDDiP, binary states	not needed	0	V	0	(c)
SLDP reverse norm	$\hat{L} \geq \text{Lip } V$	0	V	0	(a), (b)
Zhang–Sun	$l_\lambda + l_\varrho$	0, for $\mathcal{B}^R$	$V; V^R$	0; $\zeta = V(x_0) - V^R(x_0); 0$	(b)

Table 4: Oracles for the shared model: constants of [Assumption D](#) and part of [Theorem 6.3](#) used. A γ^F -greedy pass adds $\gamma^F/(1-\beta)$ to the floor. For the Jensen discretisation, L^ξ and $e(\mathcal{P})$ are as in [Proposition C.2.1](#); for SAA, $\hat{V}$ is the fixed point for the sampled laws and $\Delta^\pm$ are the biases of [Proposition D.5.1](#). For Zhang–Sun, G^k is the regularised gap, and $\mathcal{B}^R$ and V^R are the regularised operator and its fixed point.

and $(\mathbf{VG}_t)$ is [Lemma 2.3\(a\)](#). Apply [Theorem 6.3\(a\)](#) with $\kappa = 1$ and $\Gamma_k^\beta = 0$: (i) is [Lemma 6.1](#) with $e_t = 0$ and [\(33\)](#); (ii) uses the first branch of $\hat{\psi}$; for (iii), on the event of [Theorem 6.3\(a\)](#), some $k \leq \lfloor K_\varepsilon \rfloor + 1$ has $G^k \leq \varepsilon$, so $V(x_0) - \underline{V}^{k-1}(x_0) \leq \varepsilon$, and $\underline{V}^j(x_0)$ is nondecreasing in j . $\square$

Valid inexact cuts. If $\underline{V}^0 \leq V$ and every cut added to phase p is $\hat{L}$ -Lipschitz, valid for the current models and γ -tight at the trial point ($f \leq \mathcal{B}_p W_{p+1}$ on X_p and $f(x) \geq \mathcal{B}_p W_{p+1}(x) - \gamma$), then [Lemma D.1.2](#) gives [Assumption D](#) with $\gamma_{k,t} = \gamma$ and $U = V$. If moreover $\underline{V}^0$ is convex, the Jensen discretisation of [Proposition C.2.1](#), applied phase by phase to $\xi \mapsto \inf_y [\ell_p(x, y, \xi) + \beta W(y)]$ for the convex $\hat{L}$ -Lipschitz W with $\underline{V}_{p+1}^0 \leq W \leq V_{p+1}$, gives such cuts with $\gamma \leq L^\xi e(\mathcal{P})$, and the models stay convex.

Proposition D.5.1 (SAA backward passes, true-law forward passes). *Let $\hat{\mathbb{P}}_p$ be laws on Ξ_p , fixed or drawn independently of the forward noises and of the pass lengths, with periodic operators $\hat{\mathcal{B}}_p$ as in [Section 6.4](#) and fixed point $\hat{V}$. Assume [Assumption C](#) phase by phase for both laws, γ^F -greedy forward passes from the true laws, $\underline{V}_p^0$ and all cuts $\hat{L}$ -Lipschitz for a norm, $\underline{V}_p^0 \leq \hat{V}_p$, and cuts exact for $\hat{\mathcal{B}}$ and the current models W ($f \leq \hat{\mathcal{B}}_p W_{p+1}$ on X_p , with equality at the trial point). With $\mathcal{W}_p$ the set of $\hat{L}$ -Lipschitz $W : X_p \rightarrow \mathbb{R}$ with $\underline{V}_p^0 \leq W \leq \hat{V}_p$, let, as in the SAA biases [\(28\)](#), $\Delta^+ := \max_p \sup\{(\mathcal{B}_p W - \hat{\mathcal{B}}_p W)(x) : x \in X_p, W \in \mathcal{W}_{p+1}\}$ and $\Delta^- := \max_p \sup_{X_p}(\hat{\mathcal{B}}_p V_{p+1} - \mathcal{B}_p V_{p+1})$. Then, conditionally on $\hat{\mathbb{P}}$ (all statements hold under $\mathbb{P}(\cdot | \hat{\mathbb{P}})$), [Assumption D](#) holds with $U = \hat{V}$ and $\gamma_{k,t} = (\Delta^+)_+$, so [Theorem 6.5](#) applies under $(\mathbf{VG}_t)$, with $\Gamma_k^\beta = ((\Delta^+)_+ + \gamma^F)/(1-\beta)$ and $\bar{M}_U \leq \sum_p \int (\hat{V}_p - \underline{V}_p^0) d\mu_p$, and $\zeta \leq (\Delta^-)_+/(1-\beta)$. If the maps $\xi \mapsto \inf_{y \in \mathcal{X}_p(x, \xi)} [\ell_p(x, y, \xi) + \beta W(y)]$, for $x \in X_p$ and $W \in \mathcal{W}_{p+1} \cup \{V_{p+1}\}$, are L^ξ -Lipschitz on Ξ_p , then $\Delta^\pm \leq L^\xi \max_p W_1(\mathbb{P}_p, \hat{\mathbb{P}}_p)$.*

For empirical laws of N i.i.d. draws, [Proposition C.3.1](#) applies with a union bound over the S phases (the net condition holds for convex X_p by [Lemma A.1.1\(a\)](#)), the deterministic class of $\hat{L}$ -Lipschitz W with $\inf \underline{V}_{p+1}^0 \leq W \leq \bar{\ell}/(1-\beta)$, which contains $\mathcal{W}_{p+1} \cup \{V_{p+1}\}$ if $\inf \underline{V}_{p+1}^0 \leq 0$ (e.g. $\underline{V}^0 = 0$) and V_{p+1} is $\hat{L}$ -Lipschitz, and scenario values Lipschitz in x : $\Delta^\pm = O(N^{-1/(d+2)} \sqrt{\ln N})$, with constants depending on $d, D, \hat{L}$, the range in ξ and the Lipschitz constant in x of the scenario values, and only logarithmically on $\bar{\ell}/(1-\beta)$.

Proof of Proposition D.5.1. As for [Theorem 5.1](#), apply [Lemma D.1.2\(a\)](#) (periodic) to the problem with laws $\hat{\mathbb{P}}$: since $\underline{V}^0 \leq \hat{V}$ and every cut satisfies $f \leq \hat{\mathcal{B}}_p W_{p+1}$, every model satisfies $W_p \leq \hat{V}_p$, so [\(O3 \$_\infty\$ \)](#) holds with the ceiling $U = \hat{V}$, and every model of phase p lies in $\mathcal{W}_p$ by [\(O1 \$_\infty\$ \)](#). At $x = \mathbf{x}_t^k$,

the current models satisfy $W \geq \underline{V}^{k-1}$, so $f(x) = \hat{\mathcal{B}}_p W_{p+1}(x) \geq \hat{\mathcal{B}}_p \underline{V}_{p+1}^{k-1}(x) \geq \mathcal{B}_p \underline{V}_{p+1}^{k-1}(x) - \Delta^+$, because $\underline{V}_{p+1}^{k-1} \in \mathcal{W}_{p+1}$: this is (O2 $_{\infty}$) with the deterministic error $(\Delta^+)_+$. Also $\bar{U}_p \leq \hat{V}_p$. For ζ , let $e := \max_p \sup_{X_p} (\hat{V}_p - V_p)_+$. Since $\hat{\mathcal{B}}_p$ is monotone and $\hat{\mathcal{B}}_p(W + c) = \hat{\mathcal{B}}_p W + \beta c$ for constants c , $\hat{V}_p - V_p = (\hat{\mathcal{B}}_p \hat{V}_{p+1} - \hat{\mathcal{B}}_p V_{p+1}) + (\hat{\mathcal{B}}_p V_{p+1} - \mathcal{B}_p V_{p+1}) \leq \beta e + \Delta^-$, so $e \leq \beta e + (\Delta^-)_+$, that is, $\zeta \leq e \leq (\Delta^-)_+/(1 - \beta)$. For the Wasserstein bound, $(\mathcal{B}_p W - \hat{\mathcal{B}}_p W)(x)$ is the integral of an L^ξ -Lipschitz function against $\mathbb{P}_p - \hat{\mathbb{P}}_p$; use Kantorovich–Rubinstein duality. $\square$

Proposition D.5.2 (Shifted sampled cuts). *Assume [Assumption E](#) phase by phase for general laws, without (I3), with γ^F -greedy forward passes from the true laws. At $\mathbf{x}_t^k$, add the shifted sampled cut of [Appendix C.4](#) for $\mathcal{B}_{p(t)}$ and the current models, built from a deterministic number $N_{k,t}$ of fresh i.i.d. draws from $\mathbb{P}_{p(t)}$ and the scenario oracle of [Assumption F](#) (values and subgradients of norm at most $\hat{L}$ of $\inf_y [\ell_p(\cdot, y, \xi) + \beta W(y)]$ for every convex $\hat{L}$ -Lipschitz W with $\underline{V}_{p+1}^0 \leq W \leq V_{p+1}$, with range at most rg in ξ , for instance $\text{rg} \leq \bar{\ell} + \hat{L}D$), with the shift $b_{k,t} := (\text{rg} + 2\hat{L}D)\sqrt{\ln(4/\delta_{k,t})/N_{k,t}}$, $\delta_{k,t} := \delta'/(k(k+1)(t+1)(t+2))$ and $\delta' \in (0, 1]$. Let $\Gamma_k^\beta := 2 \sum_{t \geq 0} \beta^t b_{k,t} + \gamma^F/(1 - \beta)$ and $M^V := \sum_{p \in \mathbb{Z}_S} \sup_{X_p} (V_p - \underline{V}_p^0)$. There is an event A with $\mathbb{P}(A) \geq 1 - \delta'$ on which every cut is valid, and $\mathbb{E}[\mathbf{1}_A \sum_{k \geq 1} \chi_S((G^k - \Gamma_k^\beta)_+)] \leq (1 + \delta'/2) M^V$. If $N_{k,t} \geq (2(\text{rg} + 2\hat{L}D)/((1 - \beta)g))^2 \ln(4k(k+1)(t+1)(t+2)/\delta')$ for all t , then $\Gamma_k^\beta \leq g + \gamma^F/(1 - \beta)$. For $g = \varepsilon_{\text{rel}}\bar{\ell}/(1 - \beta)$ and $\text{rg} \leq \bar{\ell} + \hat{L}D$, this sample size suffices: $(2(1 + 3\hat{L}D/\bar{\ell})/\varepsilon_{\text{rel}})^2 \ln(4k(k+1)(t+1)(t+2)/\delta')$; for a β -uniform $\hat{L}$, it is independent of β except through the logarithm, and of the noise dimension.*

Proof. We follow the proof of [Theorem C.4.1](#), with $A_{k,t} := \{\sup_{X_{p(t)}} |\hat{f} - m| \leq b_{k,t}\}$ for the cut at $\mathbf{x}_t^k$, Ω_0 the whole space, $\Omega_k := \Omega_{k-1} \cap \bigcap_{t < \tau_k} A_{k,t}$ and $A := \bigcap_k \Omega_k$. On $\Omega_{k-1} \cap \bigcap_{s > t} A_{k,s}$ all cuts so far are valid ([Lemma D.1.2\(a\)](#), with $\underline{V}^0 \leq \mathcal{B}\underline{V}^0$ by (I2)), so the current models, maxima of the convex $\underline{V}^0$ and of affine $\hat{L}$ -Lipschitz cuts, lie in the class of the oracle, and on $A_{k,t}$ the shifted cut is valid and at least $\mathcal{B}_p \underline{V}_{p+1}^{k-1}(x) - 2b_{k,t}$ at $x = \mathbf{x}_t^k$, and [Lemma C.4.2](#) gives $\mathbb{P}(\Omega_{k-1} \setminus \Omega_k \mid \mathcal{A}_{k-1}) \leq \sum_{t \geq 0} \delta_{k,t} = \delta'/(k(k+1))$, so $\mathbb{P}(A^c) \leq \delta'$. In the stopped-potential argument, replace $\Psi_T((R^k - \Gamma_k)_+)$ by Λ_k of (55) with $\kappa = 1$ and $\gamma_{k,t} := 2b_{k,t}$, which is at most $\sum_p \bar{s}_p/2 \leq M^V/2$ on Ω_{k-1} (there $\underline{V}^{k-1} \leq V$, and $\hat{\psi}(u) \leq u/2$), and [Lemma 3.2](#) by the two lower bounds of the proof of [Theorem 6.5](#), which only use the definition of Λ_k : $\mathbb{E}[\mathbf{1}_{\Omega_{k-1}} \Lambda_k \mid \mathcal{A}_{k-1}] \geq \mathbf{1}_{\Omega_{k-1}} \chi_S((G^k - \Gamma_k^\beta)_+)$. Finally, $b_{k,t} \leq (1 - \beta)g/2$ for all t gives $2 \sum_t \beta^t b_{k,t} \leq g$, and this is the condition on $N_{k,t}$. $\square$

Proposition D.5.3 (Other oracles). *Let [Assumption C](#) hold, and let every cut be computed for the current model W at $x = \mathbf{x}_t^k$.*

- (a) (SDDiP.) *Let $X \subset \{0, 1\}^n$, the noise be finitely supported, the stages be mixed-integer linear with complete continuous recourse, $\underline{V}^0 = 0$, and each cut be a Lagrangian cut of $\mathcal{B}W$ with an optimal multiplier or an integer optimality cut ([Zou et al., 2019](#)), possibly with further cuts $f' \leq \mathcal{B}W$. Then [Assumption D](#) holds for $\text{dist} = +\infty$ off the diagonal, with $U = V$ and $\gamma_{k,t} = 0$, and [Theorem 6.3\(c\)](#) gives $\mathbb{E} \sum_{k \geq 1} (G^k - \gamma_k^F/(1 - \beta))_+ \leq |X|\bar{\ell}/(1 - \beta)^2 \leq 2^n \bar{\ell}/(1 - \beta)^2$.*
- (b) (Reverse-norm cuts.) *If dist is a norm, deterministic $\underline{V}^0 \leq V$, $\underline{V}^0$ and V are $\hat{L}$ -Lipschitz, and $f(y) := \mathcal{B}W(x) - \hat{L}\|y - x\|$, then [Assumption D](#) holds with $U = V$ and $\gamma_{k,t} = 0$.*
- (c) (Zhang–Sun, shared-model version.) *In the stationary setting of [Proposition B.5.1](#) (finite noise, compact X , nodal costs in $[0, \bar{\ell}]$ where finite, penalties $\varpi_\xi \mathcal{R}_\xi \geq 0$, $\mathcal{R}_\xi(0) = 0$), let the forward pass be greedy for $\mathcal{B}^R W(\bar{x}) := \int \inf_{z \in X} [\varpi_\xi \mathcal{R}_\xi(\bar{x} - z) + \dot{\mathcal{B}}(W; z, \xi)] \mathbb{P}(d\xi)$, the cut average the generalised conjugacy cuts of $\dot{\mathcal{B}}(W; \cdot, \xi)$ at $\bar{x} = \mathbf{x}_t^k$, and $\underline{V}^0 = 0$. Then the regularised problem satisfies [Assumption C](#) with $\bar{\ell} + \max_\xi \varpi_\xi \sup_{X-X} \mathcal{R}_\xi$ for $\bar{\ell}$ (given (S3 $_{\infty}$); ℓ^R below is lower semicontinuous under their assumptions), $V^R \leq V$, and [Assumption D](#) holds for $\mathcal{B}^R$,*

with G^k the regularised gap (losses ℓ^R), $\hat{L} = \sum_{\xi} \mathbb{P}(\xi)(l_{\xi,\lambda} + l_{\xi,\varrho})$, $\gamma_{k,t} = 0$ and $U = V$, so $\zeta = V(x_0) - V^R(x_0)$, or, under [Lemma B.5.2](#), $U = V^R$ and $\zeta = 0$.

Proof. (a) X and the noise are finite, so [\(S3 \$_{\infty}\$ \)](#) is not needed, every function on X is 0-Lipschitz for this metric, and $\bar{U} = U$. Validity on X and tightness at binary trial points are properties of one nodal problem, here with βW , a maximum of affine functions, as cost-to-go term: [Zou et al. \(2019, Thm. 3\)](#) for Lagrangian cuts with optimal multipliers, and their (4.2) for integer optimality cuts. So [Lemma D.1.2](#) gives [\(O2 \$_{\infty}\$ \)](#) with $\gamma_{k,t} = 0$ and [\(O3 \$_{\infty}\$ \)](#) with $U = V$, and $C_{\text{fin}} = \sum_x V(x) \leq |X|\bar{\ell}/(1-\beta)$. (b) f is continuous and $\hat{L}$ -Lipschitz, and $f(x) = \mathcal{B}W(x)$, so [\(O2 \$_{\infty}\$ \)](#) holds by [Lemma D.1.2\(b\)](#). By induction, if $W \leq V$, then $f(y) \leq \mathcal{B}V(x) - \hat{L}\|y - x\| = V(x) - \hat{L}\|y - x\| \leq V(y)$, so every model is $\leq V$. *Stage-wise absorption.* If $\mathcal{X}(\cdot, \xi)$ is constant on X and $\ell(\cdot, y, \xi)$ is c -Lipschitz for dist, comparing the infima over the common set gives $\dot{\mathcal{B}}(W; x, \xi) - \dot{\mathcal{B}}(W; x', \xi) \leq c \text{dist}(x, x')$ for $W \in \mathfrak{M}$, so $\mathcal{B}W$ and $V = \mathcal{B}V$ are c -Lipschitz, uniformly in β , and (b) applies with $\hat{L} = c$ and $\underline{V}^0 = 0$. (c) With $\ell^R(\bar{x}, y, \xi) := \inf\{\ell(z, y, \xi) + \varpi_{\xi} \mathcal{R}_{\xi}(\bar{x} - z) : z \in X, y \in \mathcal{X}(z, \xi)\}$ and $\mathcal{X}^R(\bar{x}, \xi) := \bigcup_{z \in X} \mathcal{X}(z, \xi) \subset X$, $\mathcal{B}^R$ is the operator [\(32\)](#) of the regularised problem ([Appendix B.5](#)), and $0 \leq \ell^R \leq \bar{\ell} + \varpi_{\xi} \sup_{X-X} \mathcal{R}_{\xi}$. The checks of [\(O1\)](#)–[\(O4\)](#) in the proofs of [Proposition B.5.1](#) and [Lemma B.5.2](#) concern one nodal problem; with βW as cost-to-go term, they give $\hat{L}$ -Lipschitz averaged cuts $f \leq \mathcal{B}W$ (and $f \leq \mathcal{B}^R W$ under the conditions of [Lemma B.5.2](#)) with $f(\bar{x}) \geq \mathcal{B}^R W(\bar{x})$. So [\(O1 \$_{\infty}\$ \)](#) holds, and [Lemma D.1.2](#), for $\mathcal{B}^R$ for [\(O2 \$_{\infty}\$ \)](#) and for $\mathcal{B}$ (respectively $\mathcal{B}^R$) for [\(O3 \$_{\infty}\$ \)](#), gives the rest. Finally $z = \bar{x}$ is feasible, so $\mathcal{B}^R W \leq \mathcal{B}W$, and $V^R = \mathcal{B}^R V^R \leq \mathcal{B}V^R$ gives $V^R \leq V$ by [Lemma D.1.1\(ii\)](#). $\square$

D.6 Lipschitz constants

Idea of the proof, shared model. The proof of [Proposition 6.8](#) bounds the slope of a cut of $\mathcal{B}W$ by the oscillation of $\mathcal{B}W$ on X_{η} divided by η ([Lemma D.6.1](#)), and the oscillation by optional stopping of the submartingale $\sum_{u < s} \beta^u \ell_u + \beta^s W(\mathbf{x}_s)$ ($W \leq \mathcal{B}W$). The same W appears in $\mathcal{B}W(x) \geq \beta \inf W$ and in $\mathcal{B}W(z) \leq m\bar{\ell} + \beta \inf W + \beta \hat{L}\chi$, so $\inf W$ cancels. For stage-wise functions no such cancellation is available when $m \geq 2$, and a β -uniform bound for the truncation of [Section 6.6](#) is not established here. [Remark D.6.2](#) gives examples of $(R_{m,0})$.

Lemma D.6.1 (Slopes of a cut). *Assume [\(I1\)](#) and [\(E \$_{\eta}\$ \)](#). Let $W : X \rightarrow [0, \infty)$ be convex and bounded, $x \in X$, and f affine with $f \leq \mathcal{B}W$ on X_{η} and $f(x) = \mathcal{B}W(x)$. Let $\Theta := \sup_{z \in X_{\eta}} \mathcal{B}W(z) - \mathcal{B}W(x)$. Then f is (Θ/η) -Lipschitz on $\text{aff } X$. Moreover:*

- (i) $\Theta \leq \bar{\ell} + \beta(\sup_X W - \inf_X W)$;
- (ii) under $(R_{1,0})$ ([Definition 6.7](#)), $\Theta \leq \bar{\ell}$;
- (iii) if $W \leq \mathcal{B}W$ on X , W is L_W -Lipschitz on X and $(R_{m,\chi})$ holds, then $\Theta \leq m\bar{\ell} + \beta L_W \chi$.

Proof. Lipschitz constant. Write $f = f(x) + \langle g, \cdot - x \rangle$, and let Π be the orthogonal projection onto the direction space of $\text{aff } X$. If $\Pi g \neq 0$, the point $z := x + \eta \Pi g / \|\Pi g\|$ lies in X_{η} , so $\mathcal{B}W(x) + \eta \|\Pi g\| = f(z) \leq \mathcal{B}W(z)$, that is, $\|\Pi g\| \leq \Theta/\eta$. For $y, y' \in \text{aff } X$, $|f(y) - f(y')| = |\langle \Pi g, y - y' \rangle|$. *Lower bound at x .* Losses are nonnegative and $\mathcal{X}(x, \xi) \subset X$, so $\mathcal{B}W(x) \geq \beta \inf_X W$. (i) For $z \in X_{\eta}$, $\mathcal{X}(z, \xi) \subset X$, so $\mathcal{B}W(z) \leq \bar{\ell} + \beta \sup_X W$. (ii) For $z \in X_{\eta}$, $\mathcal{X}(z, \xi) = X$, so $\mathcal{B}W(z) \leq \bar{\ell} + \beta \inf_X W$. (iii) Fix $z \in X_{\eta}$, $\epsilon_0 > 0$ and $y \in X$ with $W(y) \leq \inf_X W + \epsilon_0$. Take the policy and the stopping time of $(R_{m,\chi})$, write $\ell_s := \ell(\mathbf{x}_s, \mathbf{x}_{s+1}, \xi_{s+1})$, and let $Y_s := \sum_{u < s} \beta^u \ell_u + \beta^s W(\mathbf{x}_s)$ for $s \geq 1$. For $s \geq 1$, $\mathbf{x}_s \in X$. Since ξ_{s+1} is independent of $\xi_{[s]}$ and $\mathbf{x}_{s+1}$ is a feasible selection, $\mathbb{E}[Y_{s+1} - Y_s \mid \xi_{[s]}] \geq \beta^s (\mathcal{B}W(\mathbf{x}_s) - W(\mathbf{x}_s)) \geq 0$. So $(Y_{s \wedge \sigma})_{s \geq 1}$ is a submartingale, and optional stopping at the bounded time σ gives $\mathcal{B}W(z) \leq \mathbb{E}[\ell_0 + \beta W(\mathbf{x}_1)] = \mathbb{E}Y_1 \leq \mathbb{E}Y_{\sigma}$. Now $Y_{\sigma} \leq m\bar{\ell} + \beta^{\sigma} W(\mathbf{x}_{\sigma})$ and $W(\mathbf{x}_{\sigma}) \leq W(y) + L_W \chi$; since $W(y) + L_W \chi \geq 0$ and $\sigma \geq 1$, $\mathcal{B}W(z) \leq m\bar{\ell} + \beta(\inf_X W + \epsilon_0 + L_W \chi)$. Subtract the lower bound at x and let $\epsilon_0 \rightarrow 0$. $\square$

Proof of Proposition 6.8. By Lemma D.1.2(a), Lemma D.1.1(ii) and the convexity shown in the proof of Corollary 6.6, every current model W is convex and satisfies $0 \leq \underline{V}^0 \leq W \leq V \leq \bar{\ell}/(1-\beta)$ and $W \leq \mathcal{B}W$. (i) Lemma D.6.1(i) gives $\Theta \leq \bar{\ell} + \beta\bar{\ell}/(1-\beta) = \bar{\ell}/(1-\beta)$. (ii) By induction on the cuts, in order of creation. Let $\hat{L} := m\bar{\ell}/(\eta - \chi)$. If $\underline{V}^0$ and the cuts created before are $\hat{L}$ -Lipschitz, so is the current W , and Lemma D.6.1(iii) bounds the slope of the new cut by $(m\bar{\ell} + \beta\hat{L}\chi)/\eta \leq (m\bar{\ell} + \hat{L}\chi)/\eta = \hat{L}$. (iii) Lemma D.6.1(ii). $\square$

Periodic problems. Assume (I1)–(I3) phase by phase, with (E_η) for the data of each phase ($\mathcal{X}_p(z, \xi) \subset X_{p+1}$ for z in the neighbourhood of X_p), so that every family of models satisfies $0 \leq W_{p'} \leq \mathcal{B}_{p'}W_{p'+1}$, and let each $\underline{V}_p^0$ be L_0 -Lipschitz. Then Proposition 6.8(i) and (iii) hold phase by phase, with $(R_{1,0})$ read as $\mathcal{X}_p(z, \xi) = X_{p+1}$. For (ii), assume that, for every p , every z in the neighbourhood of X_p and every $y \in X_{p+1}$, some policy started at z at a time of phase p and a stopping time $\sigma \equiv 1 \pmod{S}$ with $1 \leq \sigma \leq m$ give $\|\mathbf{x}_\sigma - y\| \leq \chi < \eta$ almost surely, and $L_0 \leq m\bar{\ell}/(\eta - \chi)$. Then every cut is $m\bar{\ell}/(\eta - \chi)$ -Lipschitz: in the induction of Proposition 6.8(ii), apply the proof of Lemma D.6.1(iii) to $Y_s := \sum_{u < s} \beta^u \ell_u + \beta^s W_{p+s}(\mathbf{x}_s)$, a submartingale since $W_{p'} \leq \mathcal{B}_{p'}W_{p'+1}$; $\sigma \equiv 1 \pmod{S}$ makes $\inf_{X_{p+1}} W_{p+1}$ cancel. Unless one step suffices, this forces $m \geq S + 1$, a constant of the order of the loss of a period divided by η ; whether this is necessary is open.

Remark D.6.2 (Examples). *Inventory with backlog* (the newsvendor of Ju and Lan (2023, eq. (30)), with $n = 1$): state $x \in X = [-b', b]$, demand $\xi \geq 0$, next state $y \in [\max(-b', x - \xi), b]$, convex ordering, holding and backlog costs. (E_η) at $z = b + \eta$ forces $\xi \geq \eta$ a.s., so the rule $\mathbf{x}_{s+1} = \max(y, \mathbf{x}_s - \xi_{s+1})$ reaches any lower target y , going down by at least η per step, and orders reach any higher level in one step: $(R_{m,0})$ holds with $m = \lceil (b + b')/\eta \rceil + 1$. *Hydro reservoir with free spillage*: next level $y \in [0, \min(b, x + \xi)]$, inflow $\xi \geq 0$, convex thermal cost. (E_η) at $z = -\eta$ forces $\xi \geq \eta$, and the rule $\mathbf{x}_{s+1} = \min(y, \mathbf{x}_s + \xi_{s+1})$ gives $(R_{m,0})$ with $m = \lceil b/\eta \rceil + 1$. In these examples (E_η) alone gives a β -uniform bound of order $(D/\eta)\bar{\ell}/\eta$. Some neighbourhood condition is needed: $\mathcal{X}(q) = [q, 1]$ on $[0, 1]$ with loss q' gives $V(q) = q/(1-\beta)$, a Lipschitz constant of order $1/(1-\beta)$; this example violates feasibility at $q > 1$, hence (E_η) and Ju–Lan’s Assumption 2.2.

D.7 Non-stationary truncation and the warning instance

Let Pb_H be the H -stage problem of Section 2 with stage losses $\beta^{t-1}\ell$, $X_0 = \{x_0\}$, $X_t = X$ for $t \geq 1$, and $V_H = 0$. Run SDDP on Pb_H with $\underline{V}_t^0 = 0$ and exact cuts (variant (N)). Write $\mathcal{V}_t^k := \beta^{-t}\underline{V}_t^k$ for the normalised functions, let $\pi^{k,H}$ be the non-stationary greedy policy of iteration k and $G_H^k := V_0^{\pi^{k,H}}(x_0) - \underline{V}_0^{k-1}(x_0)$, with $V_0^{\pi^{k,H}}$ the H -stage cost of this policy.

Theorem D.7.1 (Truncation). Assume (I1), Assumption A for Pb_H , and that the normalised stage cuts are $\hat{L}$ -Lipschitz on X (under (E_η) , with stage cuts valid on X_η , this holds with $\hat{L} = \bar{\ell}/((1-\beta)\eta)$, and with $\hat{L} = \bar{\ell}/\eta$ under $(R_{1,0})$).

- (i) For every H and every $\varepsilon' > 0$ with $(1-\beta)\varepsilon' \leq \hat{\rho}$ ($\hat{\rho} = 4\hat{L}D$), $\mathbb{E}\#\{k \geq 1 : G_H^k > \varepsilon'\} \leq 2(d+1)\bar{\ell}\hat{\rho}^d/((1-\beta)^{d+2}\varepsilon'^{d+1})$ (Corollary 3.9).
- (ii) Let $\bar{\pi}^k$ be any policy that applies $\pi^{k,H}$ for the first H decisions and then anything feasible (for instance, restarts $\pi^{k,H}$ every H steps). Then $\underline{V}_0^{k-1}(x_0) \leq V(x_0)$ and $V^{\bar{\pi}^k}(x_0) - V(x_0) \leq V^{\bar{\pi}^k}(x_0) - \underline{V}_0^{k-1}(x_0) \leq G_H^k + \varepsilon_H$.
- (iii) Hence, for $0 < \varepsilon < \bar{\ell}/(1-\beta)$ with $(1-\beta)\varepsilon \leq \hat{\rho}$ and H as in Theorem D.4.1(ii), $\mathbb{E}\#\{k \geq 1 : V^{\bar{\pi}^k}(x_0) - \underline{V}_0^{k-1}(x_0) > \varepsilon\} \leq 2e(d+1)\bar{\ell}\hat{\rho}^d/((1-\beta)^{d+2}\varepsilon^{d+1})$, with H cuts per iteration (trial-point cuts; the optional cuts of (O4) are not counted).

Proof. (i) The assumptions of Corollary 3.9 hold, with constant $\beta^t\hat{L}$ for the cuts of index t , $\underline{V}^0 = 0$, and exact cuts. (ii) The cost incurred after time H is at most $\beta^H\bar{\ell}/(1-\beta)$. By Lemma 2.2(a),

$V_0^{k-1}(x_0) \leq \tilde{V}_0^{(H)}(x_0)$, where $\tilde{V}_0^{(H)} = \mathcal{B}^H 0 \leq \mathcal{B}^H V = V$. (iii) If $V^{\hat{\pi}^k}(x_0) - V_0^{k-1}(x_0) > \varepsilon$, then $G_H^k > \varepsilon - \varepsilon_H \geq \frac{d+1}{d+2}\varepsilon$; apply (i). The Lipschitz constants come from [Lemma D.6.1](#)(i)–(ii) applied to $W = \mathcal{V}_{t+1} \in [0, \bar{\ell}/(1-\beta)]$. $\square$

Proposition D.7.2 (Warning: the greedy policy of $\mathcal{V}_1$ can stay suboptimal). *Let $x = (u, w) \in [0, 1]^2$, $x_0 = (0, 0)$, deterministic dynamics $\mathcal{X}(u, w) = \{(1-u, w') : 0 \leq w' \leq (1+u)/2\}$, loss $\ell(x, y) = |w-u| + \alpha(1-w')$ with $0 < \alpha \leq \beta/2$, and $\bar{\ell} = 1 + \alpha$; this satisfies (I1). Run (N) with any $H \geq 2$ and $\underline{V}_t^0 = 0$, and let $\hat{\pi}^k := \pi_{\mathcal{V}_1^k}$ be the greedy stationary policy of $\mathcal{V}_1^k$ after $k \geq 0$ iterations.*

- (i) *For every exact-cut oracle whose cuts have w -slopes in $[-1, 1]$ on the slices below (for instance, subgradients of the natural extension of $\mathcal{BW}$ to $w \in \mathbb{R}$), [Corollary 3.9](#) applies ($d = 1$, $D = 1$, $\hat{L} = 1$), so almost surely $G_H^k \leq \varepsilon'$ for all large k , for every $\varepsilon' > 0$.*
- (ii) *For every exact-cut oracle that adds, at index 1, only the cut at the trial point, and every k , $\hat{\pi}^k$ is the policy $(u, w) \mapsto (1-u, (1+u)/2)$, and $V^{\hat{\pi}^k}(x_0) - V(x_0) \geq \beta(\beta-\alpha)/(1-\beta^2) \geq \beta^2/(4(1-\beta))$.*

Proof. Use the slices $X_0 = \{x_0\}$, $X_t = \{1\} \times [0, 1]$ (t odd) and $\{0\} \times [0, 1]$ ($t \geq 2$ even): convex, compact, forward invariant, with $d_t \leq 1$ and $D_t \leq 1$. (i) Cuts with w -slopes in $[-1, 1]$ are 1-Lipschitz on the slices (for the natural extension, $\mathcal{BW}(u, w) = |w-u| + \min_{0 \leq w' \leq (1+u)/2} [\alpha(1-w') + \beta W(1-u, w')]$ has w -slopes in $[-1, 1]$); apply [Corollary 3.9](#) and Borel–Cantelli (a finite expected count). (ii) By induction on k , suppose $\mathcal{V}_1^k(1, \cdot)$ is nonincreasing (true for $k = 0$). The first decision minimises $\alpha(1-w_1) + \beta \mathcal{V}_1^k(1, w_1)$ over $w_1 \in [0, 1/2]$, a strictly decreasing function, so $x_1 = (1, 1/2)$. On the slice $\{u = 1\}$, $\mathcal{BV}_2(1, w) = (1-w) + \text{const}$ is differentiable at the interior point $w = 1/2$, so an affine $f \leq \mathcal{BV}_2$ on the slice that is tight at $(1, 1/2)$ has w -slope -1 , whatever its u -slope. Hence $\mathcal{V}_1^{k+1} = \max(0, \text{cuts})$ is nonincreasing in w on every line $u = \text{const}$, which closes the induction, and at any (u, w) the objective $\alpha(1-w') + \beta \mathcal{V}_1^k(1-u, w')$ of $\hat{\pi}^k$ is strictly decreasing in w' , so $w' = (1+u)/2$. From x_0 , $\hat{\pi}^k$ alternately visits $(1, \frac{1}{2})$ and $(0, 1)$, starting with $(1, \frac{1}{2})$, with costs $\alpha/2$ at $t = 0$, $1/2$ at odd t and $1 + \alpha/2$ at even $t \geq 2$, so $(1-\beta^2)V^{\hat{\pi}^k}(x_0) = \alpha/2 + \beta/2 + \beta^2$. The policy that alternates $(0, 0) \rightarrow (1, \frac{1}{2}) \rightarrow (0, 0)$ costs $\alpha/2$ at even t and $1/2 + \alpha$ at odd t , so $(1-\beta^2)V(x_0) \leq \alpha/2 + \beta/2 + \beta\alpha$. Subtract, and use $1 + \beta \leq 2$ and $\alpha \leq \beta/2$. $\square$

D.8 Tightness of the convex-regret form

Proposition D.8.1 (Tightness of the convex-regret form). *Let $X = [0, 1]$, $x_0 = 0$, deterministic dynamics $\mathcal{X}(z) = X$ for all $z \in X_\eta$, loss $\ell(z, y) = (r_0 - az)_+ + \alpha|y - z|$ with $0 < r_0 \leq a$ and $0 < \alpha < \beta a$, and $\underline{V}^0 = 0$. Then (I1), (I2), (E $_\eta$) and (R $_{1,0}$) hold, and every cut of every possible model is $(a + \alpha)$ -Lipschitz, so $\hat{L} = a + \alpha$ and $D = 1$ are valid. Moreover (i) π^1 stays at 0 forever and $G^1 = r_0/(1-\beta)$; (ii) $V(x) = (1 + \alpha/a)(r_0 - ax)_+$ on X , so in variants (D) and (G), for every run, $\sup_k \Phi^k \leq \text{avg}_X V = (1 + \alpha/a)r_0^2/(2a)$. Consequently, if for some $\vartheta > 0$ the bound $\mathbb{E} \sum_{k \geq 1} \hat{\psi}(\vartheta G^k) \leq \text{avg}_X (V - \underline{V}^0)$ held for all instances, then $\vartheta \leq 2\sqrt{2}(1 + \alpha/a)(1-\beta)$. The same holds for any per-iteration bound $\mathbb{E}_{k-1}[\Phi^k - \Phi^{k-1}] \geq \hat{\psi}(\vartheta G^k)$.*

Proof. For every W , $\mathcal{BW}(z) = (r_0 - az)_+ + \min_{y \in X} [\alpha|y - z| + \beta W(y)]$ is the sum of an a -Lipschitz and an α -Lipschitz function on X_η , so every subgradient at a point of $X \subset \text{int } X_\eta$ has norm at most $a + \alpha$. (i) With $\underline{V}^0 = 0$, the greedy objective is $\alpha|y - x|$, uniquely minimised at $y = x$, so $x_t = 0$ and $V^{\pi^1}(0) = r_0/(1-\beta)$. (ii) Let $U(x) := (1 + \alpha/a)(r_0 - ax)_+$. For $x \geq r_0/a$, $y = x$ gives $\mathcal{BU}(x) = 0 = U(x)$. For $x < r_0/a$: on $y \in [x, r_0/a]$ the objective $\alpha(y - x) + \beta(a + \alpha)(r_0/a - y)$ is decreasing, since $\alpha < \beta(a + \alpha)$; on $y \geq r_0/a$ it is increasing, and on $y \leq x$ it is decreasing in y . So the minimum is $\alpha(r_0/a - x)$, and $\mathcal{BU}(x) = r_0 - ax + \alpha(r_0/a - x) = U(x)$. Hence $U = V$, and $\Phi^k \leq \text{avg } V$ by [Lemma D.1.2](#)(a). The average is $(1 + \alpha/a) \int_0^{r_0/a} (r_0 - as) ds$. *Consequence.* Here $\hat{\rho} = 4(a + \alpha)$, and the $k = 1$ term alone gives $\hat{\psi}(s) \leq (1 + \alpha/a)r_0^2/(2a)$ with $s := \vartheta r_0/(1-\beta)$. If

$s > \hat{\rho}$, then $\hat{\psi}(s) > \hat{\psi}(\hat{\rho}) = a + \alpha > (1 + \alpha/a)a/2 \geq (1 + \alpha/a)r_0^2/(2a)$, a contradiction. So $s \leq \hat{\rho}$, and $s^2/(4\hat{\rho}) \leq (1 + \alpha/a)r_0^2/(2a)$ gives $\vartheta^2 \leq 8(1 + \alpha/a)^2(1 - \beta)^2$. $\square$

In this instance the true potential increase is at most $\text{avg } V$, of the order of $\hat{\varphi}(r_0)$, while $G^1 = r_0/(1 - \beta)$, so *every* lower bound on the potential increase loses the factor $1 - \beta$ there, cut-by-cut accounting included.

D.9 Comparison with Ju and Lan

Ju and Lan (2023) work with an SAA of N scenarios, their λ is our β , and one iteration of their methods is one forward step and one cut; their Lipschitz constants (their Lemma 2.4) are of the order of our crude constant. With a common β -uniform constant $\hat{L}$ (Proposition 6.8 provides one under $(R_{m,\chi})$), $n = d$, $\Lambda := 4\hat{L}D/(\bar{\ell}\varepsilon_{\text{rel}})$ and $(1 - \beta)\varepsilon = \bar{\ell}\varepsilon_{\text{rel}}$, reading their Theorems 2.8, 2.11 and 3.5 with the truncation length needed for accuracy ε gives the following numbers of cuts until ε -accuracy, up to constants and norms (Inf-EDDP and Inf-SDDP are their explorative and sampling methods, and CE stands for continual exploration); all rows give an ε -tight lower bound at the root.

method	cuts until ε -accuracy	guarantee
Inf-EDDP, det. (Ju and Lan, 2023, Thm. 2.8)	$(\ln(2/\varepsilon_{\text{rel}})/(1 - \beta))^2(\Lambda + 1)^d$	first decision
CE-Inf-EDDP, det. (Ju and Lan, 2023, Thm. 2.11)	$(4\ln(2/\varepsilon_{\text{rel}})/(1 - \beta))(\Lambda + 1)^d$	first decision
CE-Inf-SDDP, random (Ju and Lan, 2023, Thm. 3.5)	$\text{CE-Inf-EDDP} \times N^{2\ln(2/\varepsilon_{\text{rel}})/(1 - \beta)}$	first decision
(G), Corollary 6.6	$2(d + 1)\Lambda^d/((1 - \beta)^2\varepsilon_{\text{rel}})$	stationary policy
(D), Theorem D.4.1	$(G) \times e\ln((d + 2)/\varepsilon_{\text{rel}})$	stationary policy
(N), Theorem D.7.1	as (D)	non-stationary policy

E Notation

X_t, d_t, D_t	state set at time t (state x_t), its dimension and diameter	$\xi_t, \mathbb{P}_t, \Xi_t$	noise of stage t , its law, its space
$\mathcal{X}_t, \ell_t, \bar{\ell}$	next-state map, stage loss, initial error per stage (18)	$\hat{\mathcal{B}}_t, \mathcal{B}_t$	pointwise (given ξ) and expected Bellman operators
$V_t, \underline{V}_t^k$	value function at time t , approximation after k iterations	f_t^k	cut added at iteration k
$\pi^k, V_0^{\pi^k}(x_0)$	greedy policy of iteration k , its expected cost	G^k	gap $V_0^{\pi^k}(x_0) - \underline{V}_0^{k-1}(x_0)$
r_t^k, R^k	residual, its sum along the path	$\gamma_t^k, \gamma_t^{F,k}$	cut error, forward error
$\mathcal{F}_{t-1}^k$	forward operator of iteration k (greedy up to $\gamma_t^{F,k}$)	Γ_k, Γ_k^F	total cut error, total forward error
$\Gamma, \bar{\Gamma}_k$	total errors (deterministic, random)	$U_t, \bar{U}_t$	ceiling, its Lipschitz envelope
M_t, M^V	potential bounds	ζ_0	ceiling excess at x_0
dist_t, L_t	metric, Lipschitz constant	μ_t, κ_t	reference measure, growth constant
$\Phi^k, \mathcal{A}_k$	potential after k iterations, σ -field of k iterations	$\bar{\mathbb{P}}_t, \Delta_t^\pm$	SAA law, SAA biases
$\varphi_t, \hat{\psi}_t, \psi_t$	lift function and its convex minorants	ρ_t, Ψ_T	$4L_t D_t$, infimal convolution
$\beta, \mathcal{B}, V$	discount, stationary operator, fixed point	$\hat{L}, \hat{\rho}$	normalised Lipschitz constant, $4\hat{L}D$
τ_k, H	pass length (random, fixed)	η, χ	recourse margin, reachability tolerance
$S, p(t)$	period, phase $t \bmod S$ of time t (Section 6.4)	χ_S, q_S	periodic lift function (53), period factor (Theorem 6.5)
$\hat{M}_t, \hat{V}_t$	SAA potential bound and value function (Section 5)	$\bar{M}_U, M_U, \bar{M}$	infinite-horizon potential bounds
Γ_k^β, ζ	infinite-horizon error floor (Section 6.3), ceiling excess at x_0	M_C, e_t	$\int(\bar{U} - \underline{V}^0) d\mu, \sup_X(\bar{U} - \underline{V}^0), \text{avg}_X(V - \underline{V}^0)$
C_U, C_{fin}	potential bounds $\sum_{t \leq T} \int(\bar{U}_t - \underline{V}_t^0) d\mu_t$ and, for finite X_t , $\sum_{t \leq T} \sum_x (U_t - \underline{V}_t^0)(x)$	$\varepsilon_H, \varepsilon_{\text{rel}}$	truncation tail $\beta^H \bar{\ell}/(1 - \beta)$, relative accuracy $((1 - \beta)\varepsilon = \bar{\ell}\varepsilon_{\text{rel}})$

Table 5: Main notation. Time and stage indices follow TFDP: stage $t \in [T]$ has loss ℓ_t and noise ξ_t (noise history $\xi_{[t]} := (\xi_s)_{s \in [t]}$), the state x_t is chosen at stage t , and $V_t, \mathcal{B}_t, \underline{V}_t^k, f_t^k, r_t^k$ are indexed by the time $t < T$ of the state they act on. Quantities with superscript k are produced in iteration k , which uses the approximations of index $k - 1$.