

ARE SINGLE-LEADER-MULTI-FOLLOWER PROBLEMS PARTICULAR CASES OF BILEVEL OPTIMIZATION?

DIDIER AUSSEL, DAVID SALAS, MARTIN SCHMIDT

ABSTRACT. We study whether single-leader-multi-follower (SLMF) problems can be reformulated as bilevel optimization problems. We consider both optimistic and pessimistic formulations in which the followers solve a generalized Nash equilibrium problem. For the optimistic case the answer is positive: we derive an equivalent bilevel problem by representing the followers' equilibria as fixed points of the solution mapping of a single optimization problem. For pessimistic problems, the situation is different: we show that every optimistic trilevel problem admits an equivalent pessimistic SLMF formulation with two followers. Conversely, for every pessimistic SLMF problem verifying a mild attainability assumption we establish an equivalent optimistic trilevel reformulation. For rational linear data, these reformulations preserve linearity, have polynomial encoding length, and can be constructed in polynomial time. Consequently, the decision problem for optimistic single-leader multi-follower problems with rational linear data is NP-complete, whereas its pessimistic counterpart is Σ_2^P -complete. Thus, unless $\text{NP} = \Sigma_2^P$, the decision problem for general pessimistic SLMF problems with rational linear data does not admit a polynomial-time, polynomial-size reduction to rational linear bilevel optimization, which reveals a potentially unexpected difference between optimistic and pessimistic SLMF problems.

1. INTRODUCTION

Bilevel optimization deals with optimization problems in which some variables are required to solve another optimization problem. These problems model hierarchical interactions between two decision makers, the leader and the follower. The leader makes a decision first and anticipates the optimal response of the follower, whose problem depends on the leader's decision. If several followers interact non-cooperatively at the lower level, one obtains a single-leader multi-follower (SLMF) problem. Each follower then solves an optimization problem that also depends on the strategies of the other followers. The resulting lower-level problem is a Nash equilibrium problem or, if the followers' feasible sets depend on the other followers' strategies, a generalized Nash equilibrium problem (GNEP). Hence, SLMF problems extend the usual single-leader single-follower (SLSF) setting of bilevel optimization.

While simply quoted by Stackelberg (1934) in his pioneering work, such combinations of hierarchical optimization and equilibrium problems have been studied in the seminal work of Pang and Fukushima (2005), and can be traced back to Sherali et al. (1983). For an overview of single- and multi-leader-follower games, including existence results, reformulations, and solution methods, we refer to Aussel and Svensson (2020). A tutorial on solving SLMF problems using SOS1 reformulations is given in Aussel et al. (2025). A central issue of these models is that the followers' equilibrium need not be unique. In the optimistic case, the leader selects the most

Date: October 13, 2026.

2020 Mathematics Subject Classification. 90Cxx, 91Axx, 68Q15, 90C33.

Key words and phrases. Bilevel optimization, Single-leader multi-follower games, Complexity theory, Optimistic vs. pessimistic.

favorable equilibrium among those satisfying the upper-level constraints. In the pessimistic case, the leader accounts for the worst equilibrium outcome and requires the coupling constraints to hold for every equilibrium. This distinction already arises in bilevel optimization (Dempe 2002; Beck et al. 2027) and is equally relevant for SLMF problems.

SLMF problems appear more difficult than their SLSF counterparts because the leader must anticipate a system of mutually dependent optimization problems. In general, minimizing the sum of the followers' objectives over their jointly feasible strategies does not characterize the equilibria of the game. Moreover, convexity of each follower's problem does not ensure convexity of the equilibrium set, and even uniqueness of each individual best response does not ensure uniqueness of the equilibrium. These properties make the analysis and solution of SLMF problems challenging. However, they do not settle whether SLMF problems are actually harder than bilevel problems. In particular, the question remains whether an SLMF problem can be reformulated as a bilevel problem while preserving properties such as convexity or linearity of the data.

In recent years, several papers have shown that problem classes that are usually studied separately can be related by suitable reformulations. Zeng (2020) develops a scheme for pessimistic bilevel optimization based on a tight optimistic bilevel relaxation and a procedure for recovering a feasible optimal solution. This allows one to apply methods for optimistic bilevel optimization to pessimistic problems. For solvable bilevel problems with rational linear data, Henke et al. (2025) show that coupling constraints can be removed while preserving linearity and the globally optimal solutions after projection onto the original variables. Henke et al. (2026) prove corresponding results for pessimistic linear bilevel problems and also derive optimistic reformulations without coupling constraints. Finally, Lefebvre et al. (2025) show that, under suitable assumptions, GNEPs can be reformulated as Nash equilibrium problems by adding one player. These reformulations preserve linear or convex structure for the respective problem classes. The common idea of these works is to use additional variables or players to obtain a formulation that allows one to apply theory and methods from another problem class.

In this paper, we study whether SLMF problems with a lower-level GNEP can be reformulated as bilevel optimization problems. We first consider the optimistic case. Inspired by the use of copied follower variables in Zeng (2020), we represent the followers' equilibria as fixed points of the solution mapping of a single optimization problem. We then introduce a copy of the followers' strategy vector at the upper level and link both copies by equality constraints. This yields an equivalent optimistic bilevel problem without any assumption on the structure of the problem. We show that the reformulation preserves convex and linear data. For explicitly encoded rational linear data, the reformulation can be constructed in polynomial time and has polynomial encoding length.

For pessimistic SLMF problems, we derive an equivalent optimistic trilevel optimization problem under an attainability assumption on truncated constraint-violation functions. The truncation allows us to handle unbounded constraint violations, and we show that the attainability assumption always holds under linear data. We also prove the reverse result: every optimistic trilevel problem can be reformulated as a pessimistic SLMF problem. To this end, we encode the feasible region of the middle- and lower-level problems through a two-player GNEP and impose optimality by robust comparisons with its equilibria. Both reformulations preserve linear data and can be constructed in polynomial time with polynomial encoding length for rational linear data. We also discuss the assumptions and modifications needed to preserve convex data.

Finally, we analyze the complexity of SLMF problems with rational linear data. Using the results of Jeroslow (1985), Buchheim (2023), and Sugishita and Carvalho (2026), we prove that the decision problem for optimistic SLMF problems with rational linear data is NP-complete, whereas its pessimistic counterpart is Σ_2^P -complete. The latter result continues to hold without coupling constraints, already for a two-follower game. We prove this by reducing the DeNegre bilevel knapsack problem, whose complexity was established by Caprara et al. (2014). We further give a linear example showing that direct exact penalization of the robust coupling constraints can fail. Thus, unless $\text{NP} = \Sigma_2^P$, there is no polynomial-time reduction of the general pessimistic SLMF decision problem with rational linear data to the corresponding optimistic linear bilevel decision problem. The optimistic and pessimistic SLMF formulations therefore have different relationships with bilevel optimization even when all data are linear. This contrast was unexpected because our starting point was the scheme of Zeng (2020) for solving pessimistic bilevel problems through optimistic bilevel optimization.

The remainder of the paper is organized as follows. Section 2 introduces the followers' equilibrium problem, the optimistic and pessimistic SLMF formulations, and the notions of convex and linear data used throughout the paper. Section 3 develops the bilevel reformulation of optimistic SLMF problems. Section 4 establishes the reformulations between pessimistic SLMF problems and optimistic trilevel problems. Section 5 derives the complexity results under rational linear data and studies the case without coupling constraints. Section 6 concludes the paper.

2. SINGLE-LEADER-MULTI-FOLLOWER PROBLEMS

For a natural number $m \in \mathbb{N}$, we write $[m] := \{1, \dots, m\}$. We consider one leader deciding the variable $x \in X \subseteq \mathbb{R}^p$ and a set $I := [N]$ of followers. Each follower $i \in I$ decides a variable $y_i \in Y_i \subseteq \mathbb{R}^{q_i}$. We write

$$Y := \prod_{i \in I} Y_i, \quad Y_{-i} := \prod_{j \in I \setminus \{i\}} Y_j, \quad q := \sum_{i \in I} q_i.$$

For $y = (y_i)_{i \in I} \in Y \subseteq \mathbb{R}^q$, the vector containing the decisions of all followers other than follower i is denoted by y_{-i} .

2.1. The Followers' Equilibrium Problem. For each $x \in X$ and $y_{-i} \in Y_{-i}$, follower i aims to minimize the objective function $f_i(x, y_i, y_{-i})$ subject to the constraint

$$y_i \in K_i(x, y_{-i}) \subseteq Y_i.$$

Thus, the followers solve the x -parametric GNEP given by the family of problems

$$\min_{y_i} \{f_i(x, y_i, y_{-i}) : y_i \in K_i(x, y_{-i})\}, \quad i \in I. \quad (1)$$

For each $x \in X$, we denote by $\text{GNEP}(x)$ the set of equilibria of (1). Equivalently, $y \in \text{GNEP}(x)$ if and only if, for every $i \in I$,

$$y_i \in K_i(x, y_{-i}) \quad \text{and} \quad f_i(x, y_i, y_{-i}) \leq f_i(x, z_i, y_{-i}) \quad \forall z_i \in K_i(x, y_{-i}).$$

We also define

$$D := \text{dom}(\text{GNEP}) = \{x \in X : \text{GNEP}(x) \neq \emptyset\}.$$

The equilibrium problem (1) is called the followers' problem.

At the end of this work, we will pay particular attention to games that are fully linear; in particular, they do not involve products of decision variables across players. These games have the players' interaction only in the constraints.

Definition 2.1 (Fully linear games). *We say that the equilibrium problem (1) is fully linear if for every $i \in I$ we have that*

- (i) $f_i(x, y_i, y_{-i}) = c_i^\top y_i$ for some vector $c_i \in \mathbb{R}^{q_i}$; and
- (ii) $K_i(x, y_{-i}) = \{y_i \in \mathbb{R}^{q_i} : A_i x + B_i y_i + C_i y_{-i} \leq b_i\} \subseteq Y_i$, for some matrices A_i, B_i, C_i and a vector b_i of appropriate dimensions.

2.2. Optimistic and Pessimistic Formulations. In the optimistic formulation, the leader aims to minimize, in terms of both x and y variables, an objective function $F(x, y)$ while satisfying the coupling constraints $G(x, y) \leq 0$, where $G = (G_1, \dots, G_m) : X \times Y \rightarrow \mathbb{R}^m$ with $m \geq 1$. The leader can select any equilibrium $y \in \text{GNEP}(x)$ to evaluate $F(x, \cdot)$ and $G(x, \cdot)$. The optimistic SLMF problem is

$$\min_{x, y} F(x, y) \quad (2a)$$

$$\text{s.t. } x \in X, \quad (2b)$$

$$G(x, y) \leq 0, \quad (2c)$$

$$y \in \text{GNEP}(x). \quad (2d)$$

In the pessimistic formulation, the equilibrium is selected adversarially. The leader must therefore account for the worst value of their objective function and must satisfy the coupling constraints for every followers' equilibrium. Using direct epigraphical reformulations as in Wiesemann et al. (2013), we can assume that the objective function of the leader is a function $\theta : X \rightarrow \mathbb{R}$ that does not depend on the followers equilibrium. Consequently, throughout the rest of the paper we consider the general pessimistic SLMF problem

$$\min_x \theta(x) \quad (3a)$$

$$\text{s.t. } x \in X \cap D, \quad (3b)$$

$$G(x, y) \leq 0, \quad \forall y \in \text{GNEP}(x), \quad (3c)$$

Note that in the pessimistic formulation, we need to make explicit the constraint $x \in D = \text{dom}(\text{GNEP})$. Otherwise, the leader might be encouraged to choose a point $x \in X$ with $\text{GNEP}(x) = \emptyset$ so that the robust constraint (3c) is satisfied by vacuity. This is not desirable in terms of modeling and so we enforce the leader to be restricted only to those decisions for which the followers' problem admits solutions. This constraint appears implicitly in the optimistic formulation.

2.3. Convexity and Linearity. We say that the followers' problem is *convex* if, for every $i \in I$ and every $(x, y_{-i}) \in X \times Y_{-i}$, the set Y_i is convex, the function

$$f_i(x, \cdot, y_{-i}) : Y_i \rightarrow \mathbb{R}$$

is convex, and the set $K_i(x, y_{-i})$ is convex.

We say that the optimistic SLMF problem (2) *has convex data* if X is convex, F and G are jointly convex, and, for every $i \in I$, the respective follower's problem is convex.

We say that the pessimistic SLMF problem (3) *has convex data* if X is convex, θ is convex, for every $i \in I$, the respective follower's problem is convex, and the functions $G_k(x, \cdot)$, $k \in [m]$, are concave, for every $x \in X \cap D$. The concavity of the robust coupling constraints is required due to the fact that there is an implicit maximization problem given by $\sup_y \{G_k(x, y) \mid y \in \text{GNEP}(x)\}$. In this maximization problem, concavity of $G_k(x, \cdot)$ is a desirable property; see e.g., Wiesemann et al. (2013).

Since an optimistic or pessimistic bilevel problem is the respective SLMF game with just one follower, the definition of a bilevel problem having convex data is included in the above definitions.

Now, if we are interested in linear formulations, we need to define linear data. We say that the optimistic SLMF problem (2) has linear data if

- (i) The leader's objective function is of the form $F(x, y) = d^\top x + e^\top y$.
- (ii) X and Y are polyhedra, i.e., sets defined by finitely many linear constraints; see, e.g., Conforti et al. (2014, Chapter 3.5).
- (iii) The coupling constraint function is of the form $G(x, y) = Tx + Wy - h$ for some matrices T, W and a vector h of appropriate dimensions.
- (iv) The lower-level GNEP is fully linear in the sense of Definition 2.1.

Similarly, we say that the pessimistic SLMF problem (3) has linear data if the leader's objective function is of the form $\theta(x) = d^\top x$ and if (ii)–(iv) hold as in the optimistic case.

Remark 2.2. In the sequel, we will refer also to linear data for multilevel problems (trilevel, in particular). The notion extends in the same way as defined for SLMF problems, and it is already a known concept in multilevel programming; see, e.g., Jeroslow (1985).

Of course, even with convex or linear data, both optimistic and pessimistic SLMF problems may be nonconvex. Indeed, even with convex data, the sets $\text{GNEP}(x)$ are not necessarily convex, and the domain $D = \text{dom}(\text{GNEP})$ might not be convex either. Furthermore, even if G is linear and $\text{GNEP}(x)$ is convex-valued, the equilibrium constraint (2d) and the robust constraint (3c) do not necessarily define convex feasible sets. The importance of convex data, at least in bilevel optimization, is that it paves the way to several approaches to either reformulate problems as single-level ones or to use different algorithmic techniques to search for local or even global optima.

3. OPTIMISTIC SLMF PROBLEMS AS BILEVEL PROBLEMS

For $(x, \hat{y}) \in X \times Y$, let $S(x, \hat{y})$ be the solution set of the optimization problem

$$\min_y \sum_{i \in I} f_i(x, y_i, \hat{y}_{-i}) \quad (4a)$$

$$\text{s.t. } y_i \in K_i(x, \hat{y}_{-i}), \quad \forall i \in I, \quad (4b)$$

$$f_i(x, y_i, \hat{y}_{-i}) \leq f_i(x, \hat{y}_i, \hat{y}_{-i}) \quad \forall i \in I. \quad (4c)$$

Lemma 3.1. *For every $x \in X$ and $y \in Y$,*

$$y \in \text{GNEP}(x) \iff y \in S(x, y).$$

Equivalently, $\text{GNEP}(x)$ is the set of fixed points of the set-valued map $S(x, \cdot) : Y \rightrightarrows Y$.

Proof. Suppose first that $y \in \text{GNEP}(x)$. Then, for every $i \in I$, one has $y_i \in K_i(x, y_{-i})$ and

$$f_i(x, y_i, y_{-i}) \leq f_i(x, z_i, y_{-i}) \quad \forall z_i \in K_i(x, y_{-i}).$$

Consequently, every feasible point z of (4) with $\hat{y} = y$ verifies $f_i(x, z_i, y_{-i}) = f_i(x, y_i, y_{-i})$ for every $i \in I$. Therefore $f_i(x, \cdot, y_{-i})$ is constant in the feasible set. In particular, y is feasible for (4) and minimizes its objective function. Hence, $y \in S(x, y)$.

Conversely, suppose that $y \in S(x, y)$ and assume, by contradiction, that $y \notin \text{GNEP}(x)$. Since y is feasible for (4), one has $y_i \in K_i(x, y_{-i})$, for every $i \in I$. Thus, for y to fail being an equilibrium, there must exist $i \in I$ and $z_i \in K_i(x, y_{-i})$ such that

$$f_i(x, z_i, y_{-i}) < f_i(x, y_i, y_{-i}).$$

Noting that the vector (z_i, y_{-i}) is feasible for (4) with $\hat{y} = y$, we can write

$$f_i(x, z_i, y_{-i}) + \sum_{j \in I \setminus \{i\}} f_j(x, y_j, y_{-j}) < \sum_{j \in I} f_j(x, y_j, y_{-j}).$$

This contradicts $y \in S(x, y)$, finishing the proof. $\square$

The fixed-point condition in Lemma 3.1 can be lifted by introducing a copy $\hat{y}$ of the followers' strategy vector, in a very similar vein to the approach of Zeng (2020).

Theorem 3.2. *The optimistic SLMF problem (2) admits the equivalent single-leader single-follower reformulation*

$$\min_{x, \hat{y}, y} F(x, y) \quad (5a)$$

$$s.t. \quad x \in X, \quad (5b)$$

$$\hat{y} \in Y, \quad (5c)$$

$$G(x, y) \leq 0, \quad (5d)$$

$$\hat{y} = y, \quad (5e)$$

$$y \in S(x, \hat{y}), \quad (5f)$$

where $S(x, \hat{y})$ is the solution set of (4). More precisely,

$$(x, y) \text{ solves (2)} \iff (x, y, y) \text{ solves (5)}.$$

Proof. The two problems have the same upper-level objective function. It is therefore enough to compare their feasible points. By Lemma 3.1,

$$\begin{aligned} (x, y) \text{ is feasible for (2)} &\iff x \in X, G(x, y) \leq 0, y \in S(x, y) \\ &\iff (x, y, y) \text{ is feasible for (5)}. \end{aligned}$$

This proves the equivalence. $\square$

The next proposition shows that the reformulation of Theorem 3.2 preserves convex/linear data and that it can be done efficiently.

Proposition 3.3. *Suppose that problem (2) has convex data. Then, the bilevel reformulation (5) has convex data as well.*

If, in addition, problem (2) has linear data, then the bilevel reformulation (5) has linear data. Moreover, under the standard explicit rational encoding of the linear data, the reformulation can be constructed in polynomial time and its encoding length is polynomially bounded by the encoding length of the original SLMF problem.

Proof. The first statement follows directly from the construction in Theorem 3.2. Indeed, for fixed $(x, \hat{y})$, the objective function in (4a) is convex in y , and the constraints in (4b) and (4c) define convex sets in y . Moreover, the upper-level constraints inherited from problem (2) are convex, the restriction $\hat{y} \in Y$ is convex, and (5e) is linear.

Now suppose that the original SLMF problem has linear data. Then the objective function in (4a) is fully linear (i.e., it is linear in all upper and lower variables), the constraints in (4b) are linear, and the optimality inequalities in (4c) reduce to

$$c_i^\top y_i \leq c_i^\top \hat{y}_i \quad \forall i \in I.$$

Hence, all upper- and lower-level objective functions and constraints of (5) are linear.

Finally, the reformulation introduces only one copy $\hat{y}$ of the followers' strategy vector and, for every follower, only copies of constraints already present in its optimization problem together with one additional linear optimality inequality. Consequently, the number of variables and constraints, as well as their encoding length, grows at most polynomially in the size of the original linear SLMF instance. $\square$

The reformulation is obtained by lifting the fixed-point condition $y \in S(x, y)$ into the two conditions $y \in S(x, \hat{y})$ and $\hat{y} = y$. The coupling constraint (5e) allows us to impose the simple upper-level restriction $\hat{y} \in Y$. This avoids adding fixed-point feasibility restrictions of the form $\hat{y}_i \in K_i(x, \hat{y}_{-i})$, which may fail to be convex constraints in the lifted upper-level variables $(x, \hat{y})$.

Remark 3.4. Theorem 3.2 states that an optimistic SLMF game can be transformed into an equivalent bilevel optimization problem. For solvable instances with rational linear data, applying Henke et al. (2025) yields a bilevel problem without coupling constraints that preserves linear data and has the same global optimal solutions after projection onto the original variables.

4. PESSIMISTIC SLMF PROBLEMS AS OPTIMISTIC TRILEVEL PROGRAMS

Our aim in this section is again to explore the potential of a bilevel-type reformulation, but now for pessimistic SLMF games. As shown by Zeng (2020), pessimistic bilevel problems can be solved, under suitable assumptions, through a tight optimistic bilevel relaxation and a subsequent recovery step. Hence, it might be natural to expect the same result with multiple followers. Thus, after Section 3 one would anticipate the possibility of reducing pessimistic SLMF problems to bilevel programming as well. However, in Section 4.1 we will show that optimistic trilevel problems, whose rational linear decision version is Σ_2^P -complete, see Jeroslow (1985) and Sugishita and Carvalho (2026), can be equivalently reformulated as pessimistic SLMF problems. Moreover, following the approach developed in Section 3, we will show in Section 4.2 that the converse also holds, meaning that any pessimistic SLMF (under a mild attainability hypothesis) can be also equivalently reformulated as an optimistic trilevel problem.

4.1. Trilevel Problems as Pessimistic SLMF Problems. Consider now the trilevel problem

$$\min_{x, y, z} F_1(x, y, z) \quad (6a)$$

$$\text{s.t. } x \in X, \quad (6b)$$

$$H_1(x, y, z) \leq 0, \quad (6c)$$

$$(y, z) \in S_1(x), \quad (6d)$$

where $S_1(x)$ is the solution set of the middle-level problem

$$\min_{y, z} F_2(x, y, z) \quad (7a)$$

$$\text{s.t. } y \in Y(x), \quad (7b)$$

$$H_2(x, y, z) \leq 0, \quad (7c)$$

$$z \in S_2(x, y), \quad (7d)$$

and $S_2(x, y)$ is the solution set of the lower-level problem

$$\min_z F_3(x, y, z) \quad (8a)$$

$$\text{s.t. } z \in Z(x, y). \quad (8b)$$

Our goal is to reduce this problem to a pessimistic SLMF problem. To do so, we first need to encode the middle and lower levels as a GNEP. To this end, fix $x \in X$ and $\hat{y} \in Y(x)$. The feasible region of problem (7)–(8) is given by

$$W_0(x) := \{(y, z) : y \in Y(x), H_2(x, y, z) \leq 0, z \in S_2(x, y)\}.$$

Let $W(x, \hat{y}) := W_0(x) \times S_2(x, \hat{y})$.

Lemma 4.1. *For every $x \in X$ and every $\hat{y} \in Y(x)$, the product $W(x, \hat{y})$ of the feasible region of the bilevel problem (7)–(8) for x together with the solution set of (8) for $(x, \hat{y})$ can be described as the equilibrium set of a two-player generalized Nash equilibrium problem.*

Proof. Consider the two-player generalized game given by

$$\begin{aligned} P_1(x, z) &:= \begin{cases} \min_y & 0 \\ \text{s.t.} & y \in Y(x), \\ & H_2(x, y, z) \leq 0, \end{cases} \\ P_2(x, y, \hat{y}) &:= \begin{cases} \min_{(z, w)} & F_3(x, y, z) + F_3(x, \hat{y}, w) \\ \text{s.t.} & z \in Z(x, y), w \in Z(x, \hat{y}). \end{cases} \end{aligned} \quad (9)$$

Then,

$$\begin{aligned} & (y, z, w) \text{ is an equilibrium for } (x, \hat{y}) \\ \iff & y \text{ solves } P_1(x, z) \text{ and } (z, w) \text{ solves } P_2(x, y, \hat{y}) \\ \iff & y \in Y(x), H_2(x, y, z) \leq 0, z \in S_2(x, y) \text{ and } w \in S_2(x, \hat{y}) \\ \iff & (y, z, w) \in W(x, \hat{y}). \end{aligned}$$

Thus, the equilibrium set of the two-player game is exactly $W(x, \hat{y})$, which completes the proof. $\square$

For each $x \in X$ and each $\hat{y} \in Y(x)$, let $W(x, \hat{y})$ be the set of equilibria of the two-player game given by (9). We have that

$$(y, z, w) \in W(x, \hat{y}) \iff (y, z) \text{ is feasible for (7)–(8) and } w \in S_2(x, \hat{y}). \quad (10)$$

Now, we propose a pessimistic SLMF problem in which the leader chooses a copy $(\hat{y}, \hat{z})$ of the middle- and lower-level variables and imposes its optimality in a robust sense against every $(y, z, w) \in W(x, \hat{y})$. With $D_W := \text{dom}(W) = \{(x, \hat{y}) : W(x, \hat{y}) \neq \emptyset\}$, this leads to the formulation

$$\min_{x, \hat{y}, \hat{z}} F_1(x, \hat{y}, \hat{z}) \quad (11a)$$

$$\text{s.t. } x \in X, \quad (11b)$$

$$\hat{y} \in Y(x), \quad (11c)$$

$$\hat{z} \in Z(x, \hat{y}), \quad (11d)$$

$$(x, \hat{y}) \in D_W, \quad (11e)$$

$$H_1(x, \hat{y}, \hat{z}) \leq 0, \quad (11f)$$

$$H_2(x, \hat{y}, \hat{z}) \leq 0, \quad (11g)$$

$$F_2(x, \hat{y}, \hat{z}) - F_2(x, y, z) \leq 0 \quad \forall (y, z, w) \in W(x, \hat{y}), \quad (11h)$$

$$F_3(x, \hat{y}, \hat{z}) - F_3(x, \hat{y}, w) \leq 0 \quad \forall (y, z, w) \in W(x, \hat{y}). \quad (11i)$$

Theorem 4.2. *The trilevel problem (6)–(8) and the pessimistic SLMF problem (11), with $W(x, \hat{y})$ being defined as the equilibrium set of the two-player game given by (9), are equivalent. More precisely, we have*

$$(x, y, z) \text{ solves (6)–(8)} \iff (x, \hat{y}, \hat{z}) = (x, y, z) \text{ solves (11)}.$$

Proof. Fix $x \in X$. First, suppose that $(x, \hat{y}, \hat{z})$ is feasible for (11). Since $(x, \hat{y}) \in \text{dom}(W)$, there exists $(y, z, w) \in W(x, \hat{y})$.

By (10), we have $w \in S_2(x, \hat{y})$. Moreover, $\hat{z} \in Z(x, \hat{y})$ and the robust constraint (11i) imply

$$F_3(x, \hat{y}, \hat{z}) \leq F_3(x, \hat{y}, w).$$

Since $w \in S_2(x, \hat{y})$ and $\hat{z} \in Z(x, \hat{y})$, the reverse inequality holds by optimality of w . Thus, we get $\hat{z} \in S_2(x, \hat{y})$. Hence, together with constraints (11c) and (11g) we get that $(\hat{y}, \hat{z})$ is feasible for problems (7)–(8). By (10), the robust constraint (11h) is equivalent to

$$F_2(x, \hat{y}, \hat{z}) \leq F_2(x, y, z)$$

for every (y, z) that is feasible for problems (7)–(8). Hence, we obtain

$$(\hat{y}, \hat{z}) \in S_1(x).$$

Conversely, suppose that $(\hat{y}, \hat{z}) \in S_1(x)$. Then $(\hat{y}, \hat{z})$ is feasible for the middle-level problem (7). In particular,

$$\hat{y} \in Y(x), \quad H_2(x, \hat{y}, \hat{z}) \leq 0, \quad \hat{z} \in S_2(x, \hat{y}) \subseteq Z(x, \hat{y}).$$

Hence, $(\hat{y}, \hat{z}) \in W_0(x)$ and $S_2(x, \hat{y}) \neq \emptyset$. Therefore,

$$W(x, \hat{y}) = W_0(x) \times S_2(x, \hat{y}) \neq \emptyset,$$

which yields $(x, \hat{y}) \in D_W$.

Now, let $(y, z, w) \in W(x, \hat{y})$. By (10), the pair (y, z) is feasible for the middle-level problem and $w \in S_2(x, \hat{y})$. Since $(\hat{y}, \hat{z}) \in S_1(x)$, we have

$$F_2(x, \hat{y}, \hat{z}) \leq F_2(x, y, z),$$

and thus the robust constraint (11h) is satisfied. Moreover, since both $\hat{z}, w \in S_2(x, \hat{y})$, they attain the same lower-level objective value, and hence

$$F_3(x, \hat{y}, \hat{z}) = F_3(x, \hat{y}, w).$$

Consequently, the robust constraint (11i) is satisfied as well.

The remaining upper-level objective and constraints coincide with those of (6) and, thus, the result follows. $\square$

Let us now study the preservation of convex data of the latter theorem. We say that Problem (6)–(7)–(8) has convex data if

- (i) The set X is convex, and the functions $F_1(\cdot)$ and $H_1(\cdot)$ are convex in (x, y, z) .
- (ii) For every $x \in X$, the set $Y(x)$ is convex and the functions $F_2(x, \cdot, \cdot)$ and $H_2(x, \cdot, \cdot)$ are convex in (y, z) .
- (iii) For every $x \in X$ and every $y \in Y(x)$, the set $Z(x, y)$ is convex and the function $F_3(x, y, \cdot)$ is convex in z .

With this definition, if the trilevel problem (6)–(7)–(8) has convex data, the robust constraints (11h) and (11i) are concave in the followers' variables (y, z, w) . However, the reformulation of Theorem 4.2 might fail to preserve convex data due to the constraints (11c)–(11d), which are not necessarily convex in $(x, \hat{y})$ and $(x, \hat{y}, \hat{z})$ respectively. This can be addressed by modifying the first player of the GNEP (9).

Let Y and Z be convex ambient sets such that $Y(x) \subseteq Y$, for every $x \in X$, and $Z(x, y) \subseteq Z$, for every $x \in X$ and $y \in Y(x)$. We regard the functions F_1, F_2, F_3, H_1, H_2 as defined on $X \times Y \times Z$, with the convexity properties stated above.

In the auxiliary game, for $\hat{y} \notin Y(x)$, define both followers' feasible sets to be empty. If $\hat{y} \in Y(x)$ but $y \notin Y(x)$, define the second follower's feasible set to be empty.

For $\hat{y} \in Y(x)$, the first player solves the problem $P_1(x, \hat{y}, \hat{z}, z)$ given by

$$\begin{aligned} \min_{(y,u,v)} \quad & 0 \\ \text{s.t.} \quad & y \in Y(x), \\ & H_2(x, y, z) \leq 0, \\ & u \in Y(x), \quad v \in Z(x, \hat{y}), \\ & u = \hat{y}, \quad v = \hat{z}. \end{aligned}$$

Define

$$\widetilde{W}(x, \hat{y}, \hat{z}) = \left\{ (y, z, u, v, w) : \begin{array}{l} (y, u, v) \text{ solves } P_1(x, \hat{y}, \hat{z}, z), \\ (z, w) \text{ solves } P_2(x, y, \hat{y}) \end{array} \right\}.$$

Then $\widetilde{W}$ encodes the constraints (11c)–(11d) as feasibility constraints: if one of them fails, P_1 becomes infeasible. Thus, (11) can be equivalently rewritten as

$$\min_{x, \hat{y}, \hat{z}} F_1(x, \hat{y}, \hat{z}) \tag{12a}$$

$$\text{s.t.} \quad (x, \hat{y}, \hat{z}) \in (X \times Y \times Z) \cap \text{dom}(\widetilde{W}), \tag{12b}$$

$$H_1(x, \hat{y}, \hat{z}) \leq 0, \tag{12c}$$

$$H_2(x, \hat{y}, \hat{z}) \leq 0, \quad \forall (y, z, u, v, w) \in \widetilde{W}(x, \hat{y}, \hat{z}), \tag{12d}$$

$$F_2(x, \hat{y}, \hat{z}) - F_2(x, y, z) \leq 0, \quad \forall (y, z, u, v, w) \in \widetilde{W}(x, \hat{y}, \hat{z}), \tag{12e}$$

$$F_3(x, \hat{y}, \hat{z}) - F_3(x, \hat{y}, w) \leq 0, \quad \forall (y, z, u, v, w) \in \widetilde{W}(x, \hat{y}, \hat{z}). \tag{12f}$$

Since nonemptiness of $\widetilde{W}(x, \hat{y}, \hat{z})$ is imposed, the robust constraint (12d) is equivalent to $H_2(x, \hat{y}, \hat{z}) \leq 0$. Its left-hand side is constant, and hence concave, in the followers' variables.

The leader's objective is convex, and the set

$$\{(x, \hat{y}, \hat{z}) \in X \times Y \times Z : H_1(x, \hat{y}, \hat{z}) \leq 0\}$$

is convex. The followers' objectives are convex in their own variables, and their feasible sets are convex under the assumptions above. Moreover, the left-hand sides of (12e) and (12f) are concave in the followers' variables. Thus, this pessimistic SLMF problem has convex data in the sense of Section 2.3.

We close this section by completing the picture of reductions with respect to linear data, as in proposition 3.3 and forthcoming proposition 4.8. We can directly formulate the following proposition.

Proposition 4.3. *Suppose that Problem (6)–(8) has linear data. Then, after removing terms independent of (z, w) from the second follower's objective in (9), the pessimistic SLMF formulation (9)–(11) has linear data.*

Moreover, under the standard explicit rational encoding of the linear data, the pessimistic SLMF reformulation can be constructed in polynomial time and its encoding length is polynomially bounded by the encoding length of the original trilevel problem.

Proof. Write

$$F_3(x, y, z) = a^\top x + b^\top y + c^\top z.$$

The second follower's objective in (9) is then

$$2a^\top x + b^\top y + b^\top \hat{y} + c^\top z + c^\top w.$$

For fixed $(x, y, \hat{y})$, removing the terms independent of (z, w) leaves the objective $c^\top z + c^\top w$ and does not change the set of minimizers. Together with the zero

objective of the first follower and the linear constraints, this yields a fully linear game in the sense of Definition 2.1.

The leader's objective and all explicit constraints in (11) are linear. Finally, the construction introduces only copies of variables and constraints from the original trilevel problem and performs a polynomial number of operations on their rational coefficients. Hence, its construction time and encoding length are polynomially bounded. The restriction to D_W is the standard equilibrium nonemptiness requirement and does not require an explicit enumeration of that set. $\square$

4.2. From Pessimistic SLMF Games to Trilevel Problems. Consider the pessimistic SLMF problem (3). For every $k \in [m]$, introduce a copy $y^k \in \mathbb{R}^q$ of the followers' strategy vector and consider

$$\begin{aligned} \max_{r_k, y^k} \quad & r_k \\ \text{s.t.} \quad & r_k - G_k(x, y^k) \leq 0, \\ & y^k \in \text{GNEP}(x). \end{aligned}$$

For fixed $x \in X \cap D$, this problem computes the maximum of $G_k(x, \cdot)$ over $\text{GNEP}(x)$ and represents it by the hypographical variable r_k . If such maximum is attained, then the k th robust constraint in (3c) can be represented by the inequality $r_k \leq 0$. However, this might not be the case. To at least rule out the problem when $G_k(x, \cdot)$ is unbounded on $\text{GNEP}(x)$ we can consider the truncated problem

$$\begin{aligned} \max_{r_k, y^k} \quad & r_k \\ \text{s.t.} \quad & r_k - G_k(x, y^k) \leq 0, \\ & r_k \leq 1, \\ & y^k \in \text{GNEP}(x). \end{aligned} \tag{13}$$

Still, the k th robust constraint in (3c) can be represented by the inequality $r_k \leq 0$ but now under the relaxed condition that the mapping $y \mapsto \min\{G_k(x, y), 1\}$ attains its maximum on $\text{GNEP}(x)$.

Since the problems in (13) are independent for different indices $k \in [m]$, they can be collected into

$$\max_{(r_k, y^k)_{k \in [m]}} \sum_{k=1}^m r_k \tag{14a}$$

$$\text{s.t.} \quad r_k - G_k(x, y^k) \leq 0, \quad \forall k \in [m], \tag{14b}$$

$$r_k \leq 1 \quad \forall k \in [m], \tag{14c}$$

$$y^k \in \text{GNEP}(x) \quad \forall k \in [m]. \tag{14d}$$

The equilibrium inclusions in (14d) are independent copies of the same followers' problem. They can therefore be reformulated simultaneously using the construction of Section 3.

Theorem 4.4. *Consider the pessimistic SLMF problem (3), and for each $k \in [m]$ define the map $\psi_k : (x, y) \mapsto \min\{G_k(x, y), 1\}$. Assume that*

$$\forall x \in X \cap D, \forall k \in [m] : \psi_k(x, \cdot) \text{ attains its maximum over } \text{GNEP}(x). \quad (\mathcal{H})$$

Then, Problem (3) admits the equivalent trilevel reformulation

$$\min_{x, (y^k, \hat{y}^k, r_k)_{k \in [m]}} \theta(x) \quad (15a)$$

$$s.t. \quad x \in X, \quad (15b)$$

$$r_k \leq 0 \quad \forall k \in [m], \quad (15c)$$

$$(y^k, \hat{y}^k, r_k)_{k \in [m]} \in S_1(x), \quad (15d)$$

where $S_1(x)$ is the solution set of the middle-level problem

$$\min_{(y^k, \hat{y}^k, r_k)_{k \in [m]}} - \sum_{k=1}^m r_k \quad (16a)$$

$$s.t. \quad r_k - G_k(x, y^k) \leq 0 \quad \forall k \in [m], \quad (16b)$$

$$r_k \leq 1 \quad \forall k \in [m], \quad (16c)$$

$$\hat{y}^k \in Y \quad \forall k \in [m], \quad (16d)$$

$$\hat{y}^k = y^k \quad \forall k \in [m], \quad (16e)$$

$$(y^k)_{k \in [m]} \in S_2(x, (\hat{y}^k)_{k \in [m]}), \quad (16f)$$

and $S_2(x, (\hat{y}^k)_{k \in [m]})$ is the solution set of the lower-level problem

$$\min_{(y^k)_{k \in [m]}} \sum_{k \in [m]} \sum_{i \in I} f_i(x, y_i^k, \hat{y}_{-i}^k) \quad (17a)$$

$$s.t. \quad y_i^k \in K_i(x, \hat{y}_{-i}^k) \quad \forall k \in [m], \forall i \in I, \quad (17b)$$

$$f_i(x, y_i^k, \hat{y}_{-i}^k) \leq f_i(x, \hat{y}_i^k, \hat{y}_{-i}^k) \quad \forall k \in [m], \forall i \in I. \quad (17c)$$

More precisely, x solves (3) if and only if there exist $(y^k, \hat{y}^k, r_k)_{k \in [m]}$ such that $(x, (y^k, \hat{y}^k, r_k)_{k \in [m]})$ solves (15)–(17).

Proof. The upper-level objective function is $\theta(x)$ in both formulations, so it is enough to prove the equivalence of their feasible leader decisions.

Suppose first that $(x, (y^k, \hat{y}^k, r_k)_{k \in [m]})$ is feasible for (15)–(17). By (16e), (16f), and the independence of the copied variables, Lemma 3.1 gives $y^k \in \text{GNEP}(x)$ for each $k \in [m]$. By (16e), (16f), and Lemma 3.1, the feasible set of the middle-level problem is the Cartesian product, over $k \in [m]$, of the feasible sets of the independent problems

$$\max_{r_k, y^k} r_k \quad s.t. \quad r_k \leq G_k(x, y^k), r_k \leq 1, y^k \in \text{GNEP}(x).$$

Since minimizing the middle-level objective $-\sum_{k=1}^m r_k$ is equivalent to maximizing $\sum_{k=1}^m r_k$, and the component problems are independent, middle-level optimality implies

$$r_k = \max_{z \in \text{GNEP}(x)} \psi_k(x, z) \quad \forall k \in [m].$$

Note that $\psi_k(x, y) \leq 0$ implies $G_k(x, y) \leq 0$. Thus, together with (15c), the above computation of r_k yields $G_k(x, y) \leq 0$, for every $y \in \text{GNEP}(x)$ and every $k \in [m]$. Therefore, x is feasible for (3).

Conversely, let x be feasible for (3). By hypothesis (H), for every $k \in [m]$ there exists $y^k \in \text{GNEP}(x)$ such that

$$\max_{z \in \text{GNEP}(x)} \psi_k(x, z) = G_k(x, y^k) \leq 0.$$

For each $k \in [m]$, set $r_k := \psi_k(x, y^k)$ and $\hat{y}^k := y^k$. By Lemma 3.1, each y^k solves a corresponding copy of the lower-level problem (4), with $(x, \hat{y}) = (x, \hat{y}^k)$. Since these

copies are independent, $(y^k)_{k \in [m]}$ solves (17), and so $(y^k, \hat{y}^k, r_k)_{k \in [m]}$ is feasible for (16). Moreover,

$$-\sum_{k=1}^m r_k = -\sum_{k=1}^m \max_{z \in \text{GNEP}(x)} \psi_k(x, z),$$

which is a lower bound for the objective value of every feasible point of the middle-level problem by (16b) and (16c). Therefore, $(y^k, \hat{y}^k, r_k)_{k \in [m]} \in S_1(x)$. Since $r_k \leq 0$ holds for every $k \in [m]$, the resulting point is feasible for (15)–(17). $\square$

Remark 4.5. Under the assumptions of Theorem 4.4, suppose that the pessimistic SLMF problem has convex data. Assume additionally that $G_k(x, \cdot)$ is concave for every $k \in [m]$ and every $x \in X$. Then, the trilevel reformulation in Theorem 4.4 has convex data. Recall that convexity here is understood with the variables of preceding levels fixed, as in the definition given after Theorem 4.2.

Remark 4.6. Of course, the constant 1 in the definition of ψ_k can be replaced with any constant $c > 0$.

The attainability hypothesis ($\mathcal{H}$) is verified for example, whenever $\text{GNEP}(\cdot)$ has compact values and the functions $G_k(x, \cdot)$ are continuous, for every $x \in X \cap D$. But also it is trivially verified whenever $\sup\{G_k(x, y) : y \in \text{GNEP}(x)\} > 1$. In general, the problem is then reduced to the case when

$$\{r \in \mathbb{R} : \exists y \in \text{GNEP}(x), r = G_k(x, y)\} \subset]-\infty, 1],$$

on which we need to verify that $G_k(x, \cdot)$ attains its maximum directly. The truncation technique reduces the verification of the attainability hypothesis ($\mathcal{H}$) to cases in which $G_k(x, \cdot)$ is bounded above on $\text{GNEP}(x)$. Under linear data, every such finite supremum is attained, as shown next.

Lemma 4.7. *Suppose that Problem (3) has linear data. Then, the hypothesis ($\mathcal{H}$) is verified.*

Proof. Fix $x \in X \cap D$ and $k \in [m]$. Since $x \in D$, $\text{GNEP}(x)$ is nonempty. Let $\alpha = \sup\{G_k(x, y) : y \in \text{GNEP}(x)\}$ which is either finite or $+\infty$. If $\alpha > 1$, there must exist $\bar{y} \in \text{GNEP}(x)$ such that $G_k(x, \bar{y}) > 1$ and so $\psi_k(x, \cdot)$ attains its maximum at $\bar{y}$. Thus, we only need to study the case in which $\alpha \leq 1$ holds.

Since the equilibrium problem (1) is fully linear, $\text{GNEP}(x)$ can be written as a finite union of nonempty polyhedra $P_1, \dots, P_s$. Indeed, for fixed x , the linear-program optimality conditions of all followers form a system of linear constraints and complementarity conditions. Resolving each complementarity condition into its two zero alternatives gives a finite union of polyhedra in the strategy and multiplier variables. Projecting out the multipliers yields the claimed representation of $\text{GNEP}(x)$. Let $t \in [s]$. Since $G_k(x, \cdot)$ is linear and

$$\sup\{G_k(x, y) : y \in P_t\} \leq \alpha < +\infty,$$

we get that there must exist $y_t \in P_t$ such that $G_k(x, \cdot)$ attains its maximum on P_t at y_t . Then, by choosing

$$\bar{y} \in \arg\max\{G_k(x, y) : y \in \{y_t : t = 1, \dots, s\}\},$$

we get that $G_k(x, \bar{y}) = \alpha$. Thus, $\psi_k(x, \cdot)$ attains its maximum on $\text{GNEP}(x)$ at $\bar{y}$, which completes the proof. $\square$

Using the lemma above and following the same vein of arguments as in Proposition 3.3, we can directly formulate the following proposition.

Proposition 4.8. *Suppose that Problem (3) has linear data. Then, the hypothesis of Theorem 4.4 holds, and the trilevel formulation (15)–(17) has linear data.*

Moreover, under the standard explicit rational encoding of the linear data, the trilevel reformulation can be constructed in polynomial time and its encoding length is polynomially bounded by the encoding length of the original pessimistic SLMF problem.

5. COMPLEXITY ANALYSIS UNDER RATIONAL LINEAR DATA

If we restrict our attention to rational linear data, then it is possible to perform complexity analysis, following the classic results from Jeroslow (1985) and more recently from Buchheim (2023) and Sugishita and Carvalho (2026).

Recall that for an optimization problem of the form $\min\{f(x) : x \in X\}$, the associated decision problem is, for a given value $t \in \mathbb{Q}$, to decide whether there exists $x \in X$ such that $f(x) \leq t$.

Theorem 5.1. *The decision problem associated to optimistic SLMF problem with rational linear data is NP-complete, while the decision problem associated to pessimistic SLMF problem with rational linear data is Σ_2^P -complete.*

Proof. Clearly, on the one hand, linear bilevel optimization is a subclass of optimistic SLMF problems with linear data. This yields that the decision problem of the latter class is NP-hard; see Jeroslow (1985). On the other hand, Theorem 3.2 and Proposition 3.3 provide a polynomial-time reduction from optimistic SLMF problems with rational linear data to rational linear bilevel optimization problems, preserving feasible objective values. Thus, the decision version of the optimistic SLMF problem with linear data is NP-complete as well, since the decision version of rational linear bilevel optimization is NP-complete; see Jeroslow (1985) and Buchheim (2023).

If one considers trilevel optimization problems with rational linear data, their decision version is Σ_2^P -complete: Σ_2^P -hardness follows from Jeroslow (1985), while Σ_2^P -completeness follows from Sugishita and Carvalho (2026).

Then Proposition 4.3 and Proposition 4.8, yield directly that the decision version of the pessimistic SLMF problem with rational linear data is Σ_2^P -complete. $\square$

In this discussion, we did not analyze pessimistic SLMF problems without coupling constraints. In general, such a problem has the form of

$$\min_{x \in X \cap D} \sup_{y \in \text{GNEP}(x)} F(x, y). \quad (18)$$

The supremum is understood in the extended-real sense and may equal $+\infty$. For a threshold $\tau \in \mathbb{Q}$, the associated decision problem asks whether there exists $x \in X \cap D$ such that $F(x, y) \leq \tau$ for every $y \in \text{GNEP}(x)$. Using the epigraphical reformulation of Wiesemann et al. (2013), this is equivalent to asking whether the following problem has a feasible point with $t \leq \tau$:

$$\min_{(x, t)} t \quad (19a)$$

$$\text{s.t. } x \in X \cap D, t \in \mathbb{R}, \quad (19b)$$

$$F(x, y) - t \leq 0 \quad \forall y \in \text{GNEP}(x). \quad (19c)$$

Under rational linear data, the epigraphical reformulation (19) is a polynomial-time reduction to the decision problem in Theorem 5.1. Consequently, the decision version of (18) belongs to Σ_2^P as a particular subclass of pessimistic SLMF problems.

However, this does not settle its hardness. It is NP-hard, since it contains pessimistic linear bilevel optimization without coupling constraints as a special case. To establish Σ_2^P -hardness of Problem (18), a first idea is to reduce problem (3) to a pessimistic SLMF problem without coupling constraints using exact penalization,

in the vein of Lefebvre and Schmidt (2025), Henke et al. (2025), and Henke et al. (2026). The idea would be to find, under linear data, a factor $\rho > 0$ large enough such that

$$\min_{x \in D \cap X} \theta(x) + \rho \sup_{y \in \text{GNEP}(x)} \sum_{k=1}^m [G_k(x, y)]_+$$

has the same solutions as (3), where $[a]_+ := \max\{0, a\}$, and then modify the equilibrium problem to replace $[G_k(x, y)]_+$ by slack variables, as in Henke et al. (2025) and Henke et al. (2026). The second step is easy to do. However, the penalization step is not always possible as the following example shows.

Example 5.2. Consider the following GNEP of four players parameterized by $x \in X = [0, 1]^2$:

$$\min_{a \in \mathbb{R}} a \quad \text{s.t.} \quad a \geq b + x_1 - 1, a \geq b - x_1, \quad (\text{P}_1)$$

$$\min_{b \in \mathbb{R}} b \quad \text{s.t.} \quad b \geq a, \quad (\text{P}_2)$$

$$\min_{c \in \mathbb{R}} c \quad \text{s.t.} \quad 0 \leq c \leq 1, c \geq d - x_2, \quad (\text{P}_3)$$

$$\min_{d \in \mathbb{R}} d \quad \text{s.t.} \quad 0 \leq d \leq 1, d \geq c. \quad (\text{P}_4)$$

Note first that at equilibrium between (P₁) and (P₂), we have that

$$a = b + \max\{x_1 - 1, -x_1\} \quad \text{and} \quad b = a.$$

This entails that $\max\{x_1 - 1, -x_1\} = 0$ and so $x_1 \in \{0, 1\}$. The equilibrium between (P₃) and (P₄) holds when

$$c = \max\{0, d - x_2\} \quad \text{and} \quad d = c.$$

Thus, $c = \max\{0, c - x_2\}$. If $x_2 = 0$, then we can choose $c = d \in [0, 1]$ and satisfy the latter equation. On the contrary, if $x_2 > 0$, then the only value that verifies the equations is $c = d = 0$. Thus, we deduce that $D = \text{dom}(\text{GNEP}) = \{0, 1\} \times [0, 1]$ and that for $x \in D$,

$$\text{GNEP}(x) = \begin{cases} \{(a, a, 0, 0) : a \in \mathbb{R}\}, & \text{if } x_2 > 0, \\ \{(a, a, c, c) : a \in \mathbb{R}, c \in [0, 1]\}, & \text{if } x_2 = 0. \end{cases}$$

Now, writing $y = (a, b, c, d)$, consider the robust constraint

$$G(x, y) = x_2 - x_1 + c \leq 0 \quad \forall y \in \text{GNEP}(x),$$

and the objective function $\theta(x) = x_1$. Then, the pessimistic SLMF problem (3) is given by

$$\begin{aligned} \min_x \quad & x_1 \\ \text{s.t.} \quad & x \in \{0, 1\} \times [0, 1], \\ & x_2 - x_1 + c \leq 0 \quad \forall c \in C(x), \end{aligned} \quad (21)$$

where $C(x)$ is the projection to the third coordinate of $\text{GNEP}(x)$, that is, $C(x) = \{0\}$ if $x_2 > 0$ and $C(x) = [0, 1]$ if $x_2 = 0$. Note that the optimal value for this problem is $\theta^* = 1$. Indeed, if $x_1 = 0$, the robust constraint is violated: for $x_2 > 0$, choose $c = 0$, and for $x_2 = 0$, choose $c = 1$. Now, take any $\rho > 0$ and consider the penalized problem

$$\min_{x \in \{0, 1\} \times [0, 1]} v(x) := x_1 + \rho \max_{c \in C(x)} [x_2 - x_1 + c]_+, \quad (\text{P}_\rho)$$

where $[x_2 - x_1 + c]_+ = \max\{0, x_2 - x_1 + c\}$. On the one hand, in the branch $\{1\} \times [0, 1]$, the robust constraint of (21) is verified, and so $v(x) = 1$. On the other

hand, in the branch $\{0\} \times [0, 1]$, we get that

$$v(x) = \begin{cases} \rho x_2, & \text{if } x_2 > 0, \\ \rho, & \text{if } x_2 = 0. \end{cases}$$

Thus, by taking $x_2 \rightarrow 0$, we deduce that the infimum value of v is 0, but it is not attained. Thus, regardless of the value of ρ , Problem (P_ρ) does not admit solutions and its infimum value is different from the optimal value of (21). $\diamond$

Even though it is not possible to reduce (3) to (18) via direct penalization of the robust constraints, we still can show that the decision version of pessimistic SLMF problems with rational linear data and no coupling constraints is Σ_2^P -hard. We reduce the DeNegre variant of the bilevel knapsack problem introduced in DeNegre (2011), which is known to be Σ_2^P -complete; see Caprara et al. (2014). For positive integers $A, B, a_1, \dots, a_n, b_1, \dots, b_n$, the DeNegre formulation is given by

$$\min_{x, y} \sum_{k=1}^n b_k y_k \quad (22a)$$

$$\text{s.t.} \quad \sum_{k=1}^n a_k x_k \leq A, \quad (22b)$$

$$x \in \{0, 1\}^n, \quad (22c)$$

$$y \in S(x), \quad (22d)$$

where $S(x)$ is the solution set of

$$\max_y \sum_{k=1}^n b_k y_k \quad (23a)$$

$$\text{s.t.} \quad \sum_{k=1}^n b_k y_k \leq B, \quad (23b)$$

$$y_k \leq 1 - x_k \quad \forall k \in [n], \quad (23c)$$

$$y \in \{0, 1\}^n. \quad (23d)$$

In this problem, each player has their private knapsacks: the leader picks some items to pack and then the follower chooses from the remaining items. The leader aims to minimize the gain of the follower.

Theorem 5.3. *The decision problem associated to pessimistic SLMF problem with rational linear data and no coupling constraints is Σ_2^P -complete.*

Proof. We only need to reduce the DeNegre bilevel knapsack problem to pessimistic SLMF problems with rational linear data. Let $A, B, a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{N}$ and consider the DeNegre bilevel knapsack problem (22)–(23).

Set $X = \{x \in \mathbb{R}^n : x \in [0, 1]^n, \sum_{k=1}^n a_k x_k \leq A\}$. Then, (22)–(23) can be written as

$$\min_{x \in X \cap \{0, 1\}^n} \max_{y \in Y(x)} \sum_{k=1}^n b_k y_k, \quad (24)$$

where $Y(x)$ is the feasible set of (23). Now, consider the two-player GNEP given by the problems

$$\begin{aligned} \min_y \quad & 0 \\ \text{s.t.} \quad & \sum_{k=1}^n b_k y_k \leq B, y \leq 1 - x, \\ & y \in [0, 1]^n, \\ & u \leq 0, v \leq 0 \end{aligned}$$

and

$$\begin{aligned} \min_{(u,v)} \quad & -\sum_{k=1}^n u_k - \sum_{k=1}^n v_k \\ \text{s.t.} \quad & u \leq 1 - y, u \leq y, \\ & v \leq 1 - x, v \leq x. \end{aligned}$$

At equilibrium, the unique best response of the second player is $u = \min\{1 - y, y\}$ and $v = \min\{1 - x, x\}$, where the minima are taken componentwise. Since $x, y \in [0, 1]^n$, these vectors are nonnegative. Thus, the first player's constraints $u \leq 0$ and $v \leq 0$ imply $u = v = 0$ and force x and y to be binary.

Conversely, for every $x \in X \cap \{0, 1\}^n$ and every $y \in Y(x)$, the tuple $(y, 0, 0)$ is an equilibrium: the first player is feasible and has constant objective, and $(0, 0)$ is the second player's unique best response. Consequently, for every $x \in X$, we have:

$$\text{GNEP}(x) = \begin{cases} \left\{ (y, 0, 0) : \begin{array}{l} \sum_{k=1}^n b_k y_k \leq B, \\ y \leq 1 - x, y \in \{0, 1\}^n \end{array} \right\}, & \text{if } x \in \{0, 1\}^n, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Since $0 \in Y(x)$ for every $x \in X \cap \{0, 1\}^n$, we obtain $D = \text{dom}(\text{GNEP}) = X \cap \{0, 1\}^n$. We can therefore rewrite (24) directly as

$$\min_{x \in X \cap D} \max_{(y,u,v) \in \text{GNEP}(x)} \sum_{k=1}^n b_k y_k,$$

which has exactly the form of a pessimistic SLMF problem with rational linear data and no coupling constraints. Since the construction is readily polynomial in the encoding of $A, B, a_1, \dots, a_n, b_1, \dots, b_n$, the proof is completed. $\square$

6. CONCLUSIONS

In this paper, we studied the relationship between SLMF problems and bilevel optimization. We showed that every optimistic SLMF problem admits an equivalent optimistic bilevel reformulation. For pessimistic SLMF problems, we established reformulations to and from optimistic trilevel optimization, with an attainability assumption required in the forward direction. Under rational linear data, this assumption always holds and the reformulations preserve linearity and have polynomial encoding length. The associated decision problems are NP-complete in the optimistic case and Σ_2^P -complete in the pessimistic case, even without coupling constraints and with only two followers.

These results clarify when methods for bilevel optimization can be applied to SLMF problems and when an additional optimization level is needed. A natural next step would be to investigate the computational performance of the proposed reformulations. It is also of interest to identify subclasses of pessimistic SLMF problems that admit reformulations as linear bilevel problems despite the complexity obstruction for the general class.

ACKNOWLEDGEMENTS

The second author was partially supported by CMM BASAL Grant FB210005 and FONDECYT Grant 1251159 (ANID-Chile).

USE OF AI

OpenAI's GPT was used to improve writing and presentation of the manuscript, and it helped for preliminary checking of correctness of the proofs. All the mathematical content (statements and proofs) are due to the authors, with the exception of Example 5.2, which was obtained using GPT.

REFERENCES

- Aussel, D., C. Egea, and M. Schmidt (2025). “A tutorial on solving single-leader-multi-follower problems using SOS1 reformulations.” In: *Int. Trans. Oper. Res.* 32.3, pp. 1227–1250. DOI: [10.1111/itor.13466](https://doi.org/10.1111/itor.13466).
- Aussel, D. and A. Svensson (2020). “A short state of the art on multi-leader-follower games.” In: *Bilevel optimization. Advances and next challenges*. Cham: Springer, pp. 53–76. DOI: [10.1007/978-3-030-52119-6_3](https://doi.org/10.1007/978-3-030-52119-6_3).
- Beck, Y., I. Ljubić, and M. Schmidt (2027). *Linear and Mixed-Integer Bilevel Optimization: Theory and Algorithms*. Forthcoming; pre-publication version available online. Cambridge University Press. DOI: [10.1017/9781009862318](https://doi.org/10.1017/9781009862318). URL: <https://yasminebeck.github.io/files/bilevel-optimization-cup.pdf>.
- Buchheim, C. (2023). “Bilevel linear optimization belongs to NP and admits polynomial-size KKT-based reformulations.” In: *Oper. Res. Lett.* 51.6, pp. 618–622. DOI: [10.1016/j.orl.2023.10.006](https://doi.org/10.1016/j.orl.2023.10.006).
- Caprara, A., M. Carvalho, A. Lodi, and G. J. Woeginger (2014). “A study on the computational complexity of the bilevel knapsack problem.” In: *SIAM J. Optim.* 24.2, pp. 823–838. DOI: [10.1137/130906593](https://doi.org/10.1137/130906593).
- Conforti, M., G. Cornuéjols, and G. Zambelli (2014). *Integer programming*. Vol. 271. Grad. Texts Math. Cham: Springer. DOI: [10.1007/978-3-319-11008-0](https://doi.org/10.1007/978-3-319-11008-0).
- Dempe, S. (2002). *Foundations of Bilevel Programming*. Vol. 61. Nonconvex Optimization and Its Applications. New York: Springer. DOI: [10.1007/b101970](https://doi.org/10.1007/b101970).
- DeNegre, S (2011). “Interdiction and Discrete Bilevel Linear Programming.” PhD thesis. Lehigh University, Bethlehem, PA.
- Henke, D., H. Lefebvre, M. Schmidt, and J. Thürauf (2025). “On coupling constraints in linear bilevel optimization.” In: *Optim. Lett.* 19.3, pp. 689–697. DOI: [10.1007/s11590-024-02156-3](https://doi.org/10.1007/s11590-024-02156-3).
- (2026). “On coupling constraints in pessimistic linear bilevel optimization.” In: *J. Optim. Theory Appl.* 210.2. Id/No 25, p. 12. DOI: [10.1007/s10957-026-03026-x](https://doi.org/10.1007/s10957-026-03026-x).
- Jeroslow, R. G. (1985). “The polynomial hierarchy and a simple model for competitive analysis.” In: *Math. Program.* 32, pp. 146–164. DOI: [10.1007/BF01586088](https://doi.org/10.1007/BF01586088).
- Lefebvre, H., D. Salas, and M. Schmidt (2025). *A One-Extra Player Reduction of GNEPs to NEPs*. URL: <https://optimization-online.org/?p=32565>.
- Lefebvre, H. and M. Schmidt (2025). *Exact Augmented Lagrangian Duality for Nonconvex Mixed-Integer Nonlinear Optimization*. URL: <https://optimization-online.org/?p=27046>.
- Pang, J.-S. and M. Fukushima (2005). “Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games.” In: *Comput. Manag. Sci.* 2.1, pp. 21–56. DOI: [10.1007/s10287-004-0010-0](https://doi.org/10.1007/s10287-004-0010-0).
- Sherali, H. D., A. L. Soyster, and F. H. Murphy (1983). “Stackelberg-Nash-Cournot equilibria: Characterizations and computations.” In: *Oper. Res.* 31, pp. 253–276. DOI: [10.1287/opre.31.2.253](https://doi.org/10.1287/opre.31.2.253).

- Stackelberg, H. von (1934). *Marktform und Gleichgewicht*. Berlin: Springer.
- Sugishita, N. and M. Carvalho (2026). *Decision Problems in Multilevel Linear Programming*. arXiv: [2605.04929](https://arxiv.org/abs/2605.04929) [math.OC]. URL: <https://arxiv.org/abs/2605.04929>.
- Wiesemann, W., A. Tsoukalas, P.-M. Kleniati, and B. Rustem (2013). “Pessimistic bilevel optimization.” In: *SIAM J. Optim.* 23.1, pp. 353–380. DOI: [10.1137/120864015](https://doi.org/10.1137/120864015).
- Zeng, B. (2020). “A practical scheme to compute the pessimistic bilevel optimization problem.” In: *INFORMS J. Comput.* 32.4, pp. 1128–1142. DOI: [10.1287/ijoc.2019.0927](https://doi.org/10.1287/ijoc.2019.0927).
- (D. Aussel) LAB. PROMES UPR CNRS 8521, UNIVERSITY OF PERPIGNAN VIA DOMITIA, TECNODUD, PERPIGNAN 66100, FRANCE
Email address: aussel@univ-perp.fr
- (D. Salas) INSTITUTO DE CIENCIAS DE LA INGENIERÍA, UNIVERSIDAD O’HIGGINS, RANCAGUA, CHILE
Email address: david.salas@uoh.cl
- (M. Schmidt) TRIER UNIVERSITY, DEPARTMENT OF MATHEMATICS, UNIVERSITÄTSRING 15, 54296 TRIER, GERMANY
Email address: martin.schmidt@uni-trier.de