Exact Equilibrium Analysis of Sparse Network Games in Subexponential Time

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Abstract

In October 2026 two independent groups claimed deterministic algorithms that count the satisfying assignments of a k-CNF formula in time $2^{o(n)}$, refuting the Exponential Time Hypothesis (ETH). We ask what these claims imply for the exact analysis of equilibria in games where each player interacts with a bounded number of others. Conditional on the claimed algorithm of Claude (Anthropic), we show that for such sparse network games a single object—the *equilibrium generating function*, which records every pure Nash equilibrium together with its welfare—can be computed in time $2^{O(N/\sqrt{\log N})}$ for N players. From it one reads off, exactly, the number of pure equilibria, their existence and uniqueness, the best and worst equilibrium welfare (hence the price of anarchy and the price of stability), equilibrium-selection values, and the optimistic and pessimistic values of a Stackelberg leader with a polynomial list of commitments. We complement the algorithm with hardness results for games with two actions per player and payoffs that depend on at most two other players: beyond the known NP-completeness of existence and #P-completeness of counting, uniqueness, "PoS = 1", "PoA = 1", selection values and the optimal pessimistic commitment are NP- or coNP-hard even when an equilibrium is guaranteed to exist. The reductions are linear, so under the Stearns–Hunt hypothesis none of these problems can be solved in time $2^{N^{1-\varepsilon}}$; conditional on the engine, the exponent is therefore $N^{1-o(1)}$. We illustrate the results on capacity procurement games in which buyers choose among nearby suppliers and delimit what the approach cannot do. All results are asymptotic.

1 Introduction

Many questions about equilibria in games with many players are computationally hard. Deciding whether a game in which each player's payoff depends on a few neighbours has a pure Nash equilibrium is NP-complete [13], already with two actions per player and payoffs depending on two neighbours, where counting pure equilibria is #P-complete [12]. Computing a leader's optimal commitment when followers play a Nash equilibrium is NP-hard and, in its pessimistic form, hard to approximate [5, 7, 8]. Behind such results, and behind the expectation that no algorithm can be much faster than exhaustive search, stands the Exponential Time Hypothesis [14]: 3-SAT on n variables cannot be solved in time $2^{o(n)}$. Through linear-size reductions, ETH implied that none of the equilibrium problems above could be solved in time $2^{o(N)}$.

In October 2026 two independent papers claimed to refute ETH. The first, by OpenAI [19], gives a deterministic algorithm counting the solutions of a k-CNF in time $(n+m)^{O_k(1)}2^{O_k(n\log\log n/\sqrt{\log n})}$ and polynomial space. The second, by Claude (Anthropic) [6], gives a deterministic algorithm with time $2^{O(\min\{k^2 n\log\log n/\sqrt{\log n},\, km/\sqrt{\log m}\})}$, based on rewriting weighted pairwise networks as signed sums of networks with fewer shared bits. Both come with Lean formalisations of versions of their main statements; neither had been independently refereed when this paper was written.

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All algorithmic results below are conditional on these claims; the hardness and tightness results are not.

Question. If ETH is false, which equilibrium problems actually become easier, and by how much? A refutation of ETH does not by itself speed up any other problem: a running time of $2^{o(n)}$ survives only reductions that keep the number of variables linear. The question is therefore one of encoding: which game problems can be written as weighted constraint networks of linear size, and what can be computed from such networks?

Contributions.

1. **Transfer lemma (Section 3).** The engine of Claude [6] evaluates arbitrary integer-weighted pairwise networks; only the construction of the network from a CNF is specific to SAT. We record the version for weighted constraint satisfaction problems of bounded arity (Lemma 1). This is bookkeeping on top of Claude [6], but it is the form we need.

2. **Equilibrium generating function (Section 4).** For network games with N players, actions encoded in b bits and each payoff depending on at most k players, $G(z) = \sum_{a \in \mathrm{PSNE}} z^{\mathrm{SW}(a)}$ can be computed in time $2^{O(kbN/\sqrt{\log N})}\mathrm{poly}(\cdot)$ (Theorem 3). One computation yields counting, uniqueness, exact price of anarchy and stability, selection values, and optimistic and pessimistic Stackelberg values (Corollaries 4–7); the same holds for Bayesian network games with a constant number of types (Proposition 8). Under ETH none of these problems could have been solved in time $2^{o(N)}$.

3. **Hardness with guaranteed existence (Section 5).** For two actions and payoffs depending on at most two other players—the smallest neighbourhood for which existence is NP-hard [12]—we give a parsimonious reduction and a switch construction showing that uniqueness, "PoS = 1", "PoA = 1", selection values and the optimal pessimistic commitment (with a commitment that changes the followers' game) are NP- or coNP-hard even when an equilibrium is guaranteed to exist (Theorem 11).

4. **Tightness (Section 6).** The reductions are linear. Under the Stearns–Hunt hypothesis SH, shown by MIT Hardness Group et al. [18] to be equivalent to Weak ETH, none of these problems admits a $2^{N^{1-\varepsilon}}$-time algorithm (Theorem 13); conditional on the engine, their exponent is $N^{1-o(1)}$.

5. **Illustration and limits (Sections 7–8).** For capacity procurement games in which buyers choose one of a few nearby suppliers, all of the above apply; with a single shared supplier the same counting and welfare questions are polynomial (Proposition 14). We also delimit what the method cannot do.

Related work. Gottlob et al. [13] show that existence of pure Nash and of Pareto Nash equilibria is NP-complete, that existence of strong Nash equilibria is Σ_2^P-complete and that checking whether a given equilibrium is Pareto optimal is coNP-complete; relating equilibria to constraint satisfaction, they find tractable classes with small neighbourhoods and bounded hypertree width. Fischer et al. [12] sharpen existence hardness to two actions, two-bounded neighbourhoods and two-valued payoffs, by a reduction from circuit satisfiability whose equilibria correspond one-to-one to satisfying assignments, so that counting is #P-complete; with one-bounded neighbourhoods existence is NL-complete. Daskalakis and Papadimitriou [11] reduce pure equilibria in graphical games to

inference in Markov random fields; our generating function is a weighted partition function of this kind, and what is new is that it can be evaluated in subexponential time without any width assumption. Jiang and Safari [15] show that, for recursively enumerable classes of bounded in-degree graphs, existence is polynomial-time solvable if and only if the graphs obtained by iterated removal of sinks have bounded treewidth (under standard assumptions); our results concern the classes on the hard side of this dichotomy and give a subexponential exact algorithm for them. Lampis and Lu [16] give width-parameterised algorithms and matching lower bounds under the primal-pathwidth SETH; their parameters and ours are complementary. On commitment, Conitzer and Sandholm [9] initiated the complexity analysis of optimal commitment; Coniglio et al. [7], Coniglio et al. [8] and Castiglioni et al. [5] prove hardness and inapproximability with equilibrium-playing followers; in particular, for normal-form games with a mixed-strategy leader and pure-strategy followers, the optimistic problem is polynomial while the pessimistic one is NP-hard with two followers and inapproximable with three or more [8]. Those results concern games given in normal form, whose size is exponential in the number of followers; ours concern succinct network games with many followers. Basilico et al. [2] give bilevel programming formulations; existing exact methods take exponential time in the worst case. MIT Hardness Group et al. [18] develop lower bounds under SH after the refutation of ETH; they treat classical problems in complexity theory but not games. Capacity allocation games go back to Cachon and Lariviere [4]; sourcing decisions in multitier supply networks under disruption risk are studied by Bimpikis et al. [3]. Existence of pure equilibria in singleton congestion games with player-specific payoffs is due to Milchtaich [17]; Ackermann et al. [1] show that weighted and player-specific congestion games have pure equilibria when strategy spaces are bases of matroids, and that this is the maximal such property that ignores how strategy spaces interweave.

A caveat on practice. The speed-ups are asymptotic. Claude [6] estimate that their algorithm beats exhaustive search on 3-SAT with $m = 5n$ clauses only for $n > 10^{10000}$. Nothing here suggests that equilibria of practical games become easier to compute. The contribution is a map of where exact equilibrium analysis sits once ETH is gone.

2 Model and preliminaries

Network games. A *network game* Γ has players $[N]$, finite action sets A_i with $|A_i| \le 2^b$, and integer payoff functions $u_i : \prod_j A_j \to \mathbb{Z}$. For each player i there is a set $H(i) \ni i$ such that $u_i(a)$ depends only on $a_{H(i)}$; the *interaction bound* is $k = \max_i |H(i)|$. Payoffs are given as tables over $\prod_{j \in H(i)} A_j$, or by an oracle evaluated in time τ. Let $U_{\max} = \max_{i,a} |u_i(a)|$.

A profile a is a *pure Nash equilibrium* if $u_i(a) \ge u_i(a'_i, a_{-i})$ for all i and $a'_i \in A_i$; $\mathrm{PSNE}(\Gamma)$ denotes the set of such profiles. Since $u_i(a'_i, a_{-i})$ depends only on $(a'_i, a_{H(i)\setminus\{i\}})$, the best-response indicator

$$\beta_i(a_{H(i)}) = \mathbf{1}\big[u_i(a) \ge u_i(a'_i, a_{-i})\ \forall a'_i \in A_i\big]$$

is a function of $a_{H(i)}$. Welfare is $\mathrm{SW}(a) = \sum_i u_i(a)$ and $\mathrm{OPT} = \max_a \mathrm{SW}(a)$. When $\mathrm{PSNE} \neq \emptyset$ and welfare is positive, $\mathrm{PoA} = \mathrm{OPT}/\min_{a \in \mathrm{PSNE}} \mathrm{SW}(a)$ and $\mathrm{PoS} = \mathrm{OPT}/\max_{a \in \mathrm{PSNE}} \mathrm{SW}(a)$.

Local objectives, selection and commitment. A *local objective* is $f(a) = \sum_i f_i(a_{H(i)})$ with integer terms $|f_i| \le F_{\max}$ (one term per player, on that player's neighbourhood). The optimistic and pessimistic *selection values* are $\max_{a \in \mathrm{PSNE}} f(a)$ and $\min_{a \in \mathrm{PSNE}} f(a)$. A *Stackelberg leader* chooses ℓ from a finite list L; each ℓ determines a follower game Γ_ℓ with the same sets $H(i)$ (only table entries change) and a local leader objective f_ℓ. The optimistic (pessimistic) value of ℓ is the

maximum (minimum) of f_ℓ over $\mathrm{PSNE}(\Gamma_\ell)$, with a fixed convention when $\mathrm{PSNE}(\Gamma_\ell) = \emptyset$; the leader picks ℓ maximising this value.

Hypotheses. ETH: 3-SAT on n variables has no $2^{o(n)}$-time algorithm [14]. Following MIT Hardness Group et al. [18], the Stearns–Hunt hypothesis SH states that CNF-SAT on inputs of length L (counting variables, clauses and literal occurrences) cannot be solved in time $O(2^{L^{1-\varepsilon}})$ for any $\varepsilon > 0$, and Weak ETH states that 3-SAT has no $O(2^{n^{1-\varepsilon}})L^{O(1)}$-time algorithm for any $\varepsilon > 0$. MIT Hardness Group et al. [18] show that the two are equivalent, and that a polynomial-time reduction mapping 3-SAT formulas of size $s = \Theta(n+m)$ to instances of size $N \le s^{\gamma+o(1)}$ rules out $O(2^{N^{1/\gamma-\varepsilon}})$-time algorithms under SH. A counting analogue, #SH, is treated in the same paper.

The engines.

Fact 1 (19)**.** *For each fixed $k \ge 2$ there is a deterministic algorithm that counts the satisfying assignments of a k-CNF with n variables and m clauses in time $(n+m)^{O_k(1)}2^{O_k(n\log\log n/\sqrt{\log n})}$ and polynomial space.*

Fact 2 (6, Definitions 1–2, Lemma 8 and Theorem A)**.** *A* network *on nodes $[g]$ consists of setting sets $S_a \subseteq \{0,1\}^{P_a}$, integer weights W_a on S_a, and integer pairwise factors $\Psi_{a,a'}$, each factoring through $c_{a,a'}$ bits, all values bounded by 2^B in absolute value and computable in time τ; its value is $Z = \sum_z \prod_a W_a(z_a) \prod_{a<a'} \Psi_{a,a'}(z_a, z_{a'})$. One deterministic procedure, the same for all networks, computes Z. With $g = 3K^8$, $P \ge \max_a\{P_a, \sum_{a'} c_{a,a'}\}$ and $U = \sum_{(a,a')} c_{a,a'}$ (over ordered pairs), its cost is bounded as in equation (45) of Claude [6]; for the network built from a k-CNF in their Section 6 it is $2^{O(km/\sqrt{\log m})}$.*

3 A transfer lemma for weighted constraint networks

A *weighted CSP* $\mathcal{I}$ has variables $x_1, \dots, x_N$ with domains D_v, $|D_v| \le 2^b$, and m constraints; constraint e has scope I_e with $|I_e| \le k$ and an integer table φ_e with $|\varphi_e| < 2^\beta$, each entry computable in time τ. Its value is $Z(\mathcal{I}) = \sum_x \prod_e \varphi_e(x_{I_e})$. We assume every variable lies in some scope; a free variable contributes a factor $|D_v|$.

Lemma 1 (Transfer lemma)**.** *Conditional on Fact 2, there is a deterministic algorithm computing $Z(\mathcal{I})$ in time*

$$2^{O(kbm/\sqrt{\log m})} \cdot \mathrm{poly}(N, m, \beta, \tau),$$

with an absolute constant in $O(\cdot)$.

Proof. Network. As in Claude [6, Section 6], call a pair (e, v) with $v \in I_e$ an *occurrence*. For each variable, list its occurrences $o_1, \dots, o_d$ and introduce equality tests $\{o_t, o_{t+1}\}$, each on b bits. Fix $K = K_\ell$, the largest value in the recursion of Claude [6] with $K_\ell \le m^{1/9}$, and $g = 3K^8 \le m$. Partition the constraints into g clusters of at most $\lceil m/g \rceil$ constraints. The setting of a cluster assigns a b-bit value to each of its occurrences. Its weight W_a is the product of the tables φ_e of its constraints at those values, times the indicators of the equality tests inside the cluster, and 0 on strings outside $\prod D_v$. The factor $\Psi_{a,a'}$ is the conjunction of the tests between a and a'; by Claude [6, Definition 1(b)] it factors through b times the number of such tests bits. Each occurrence lies in at most two tests, so $P = 2kb\lceil m/g \rceil$ bounds both the setting length and the shared bits of a cluster; the total number of shared bits is $U < 2kbm$; weights are bounded by $2^{B_{\mathrm{net}}}$ with $B_{\mathrm{net}} = \lceil m/g \rceil \beta$;

and W_a is computable in time $\mathrm{poly}(m,\beta)\tau$. Each test is checked exactly once, so the nonzero terms of the network correspond to assignments x and contribute $\prod_e \varphi_e(x_{I_e})$; the network value is $Z(\mathcal{I})$.

Cost. Apply Claude [6, Lemma 8, case (i), and Theorem A(b)] exactly as in their proof of Theorem A(c), replacing km by kbm and their $B = 1$ by B_{net}. The logarithm of the operation count is at most

$$2e_\ell\, kbm + O\!\left(\frac{kbm}{K}\right) + O(K^8 \log K) + O\big(K^2 \log(kbm\beta)\big),$$

where $e_\ell = O((\log m)^{-1/2})$ by Claude [6, Theorem A(a)] with the pieces of their Theorem B ($\alpha' = 1/2$), and $K \geq m^{1/90}$ because $K_{\ell+1} \leq K_\ell^{10}$. The second and third terms are $O(kbm \cdot m^{-1/90})$ and $O(m^{8/9} \log m)$. The last term comes from the factor $(\lambda + B')^{C_0 K^2}$ of Theorem A(b); it is subexponential in m but not polynomial in β. To remove the dependence on β, run the algorithm modulo each of $\mathrm{poly}(m\beta + Nb)$ small primes p, with tables reduced modulo p (so $\beta' = O(\log p)$), and recover Z by the Chinese remainder theorem. For m below the absolute threshold of Claude [6], use exhaustive search, which is within the bound because m is bounded. □

Remark 2. The lemma records that the construction of Claude [6, Section 6] never uses the fact that the constraints are clauses. Sparsification is not needed for networks that are sparse to begin with. An alternative route uses Fact 1, encoding each constraint and its weight by a linear-size circuit; for counting it gives the same conclusions with a weaker bound and polynomial space.

4 The equilibrium generating function

For a network game Γ with integer payoffs, set $s_0 = NU_{\max}$ and

$$G_\Gamma(z) = \sum_{a \in \mathrm{PSNE}(\Gamma)} z^{\mathrm{SW}(a)+s_0}.$$

G_Γ is a polynomial of degree at most $D = 2NU_{\max}$ with nonnegative integer coefficients: the coefficient of z^{w+s_0} is the number of pure equilibria with welfare w.

Theorem 3. *Conditional on Fact 2, let Γ be a network game with N players, actions encoded in b bits, interaction bound k and integer payoffs bounded by $U_{\max}$. Then G_Γ can be computed in time*

$$2^{O(kbN/\sqrt{\log N})} \cdot \mathrm{poly}(N, U_{\max}, 2^{kb}, \tau).$$

In particular, for constant k, b and $U_{\max} = \mathrm{poly}(N)$ the time is $2^{O(N/\sqrt{\log N})}$.

Proof. Fix an integer $z \geq 0$. Consider the weighted CSP with variables $a_1, \ldots, a_N$ (domains A_i) and one constraint per player i with scope $H(i)$ and table

$$\varphi_i(a_{H(i)}) = \beta_i(a_{H(i)}) \cdot z^{u_i(a)+U_{\max}}.$$

Its value is $\sum_a \prod_i \beta_i\, z^{u_i(a)+U_{\max}} = \sum_{a \in \mathrm{PSNE}} z^{\mathrm{SW}(a)+s_0} = G_\Gamma(z)$. There are $m = N$ constraints of arity at most k; each entry is computed by comparing at most 2^b payoffs, and $|\varphi_i| \leq z^{2U_{\max}}$, so $\beta = O(U_{\max} \log(z+2))$. By Lemma 1, $G_\Gamma(z)$ is computed in the stated time. Evaluating at $z = 0, 1, \ldots, D$ and interpolating recovers G_Γ. (A single evaluation at $z = 2^{Nb+1}$ also suffices, since all coefficients are below 2^{Nb}.) □

Corollary 4 (Existence, counting, uniqueness)**.** *Under the conditions of Theorem 3, the number of pure Nash equilibria, and hence existence and uniqueness, can be determined in time*

$$2^{O(kbN/\sqrt{\log N})} \cdot \mathrm{poly}(N, 2^{kb}, \tau).$$

Counting does not need integer payoffs: take $z = 1$ and compute β_i in exact rational arithmetic.

Corollary 5 (Exact price of anarchy and stability)**.** *Under the conditions of Theorem 3, the best and worst equilibrium welfare are the largest and smallest exponents of G_Γ with nonzero coefficient, minus s_0.* OPT *is obtained in the same time by dropping the factors β_i. Hence* PoA *and* PoS *can be computed exactly whenever they are defined.*

Corollary 6 (Selection values)**.** *For any local objective f as in Section 2, the optimistic and pessimistic selection values, and the distribution of f over* PSNE*, can be computed in time*

$$2^{O(kbN/\sqrt{\log N})} \cdot \mathrm{poly}(N, F_{\max}, 2^{kb}, \tau),$$

by replacing u_i with f_i in the exponent of φ_i.

Corollary 7 (Stackelberg commitment)**.** *For a leader with commitment list L, the optimistic and pessimistic values of every $\ell \in L$, and an optimal commitment under either convention, can be computed in time*

$$|L| \cdot 2^{O(kbN/\sqrt{\log N})} \cdot \mathrm{poly}(N, F_{\max}, 2^{kb}, \tau).$$

Proof. Apply Corollary 6 to (Γ_ℓ, f_ℓ) for each $\ell \in L$ and compare. □

Proposition 8 (Bayesian network games)**.** *Consider a Bayesian network game in which player i has a type t_i from a set T_i with $|T_i| \le T$, types are independent with rational prior, and u_i depends on $(a_{H(i)}, t_{H(i)})$. A pure strategy is a map $s_i : T_i \to A_i$, encoded in Tb bits. Conditional on Fact 2, the conclusions of Corollaries 4–7 hold for pure Bayes–Nash equilibria (interim, with positive priors), with b replaced by Tb, provided expected payoffs are scaled to integers bounded by* $\mathrm{poly}(N)$ *for the welfare-based statements.*

Proof sketch. By independence of types, the interim best-response condition of i at type t_i is an expectation of u_i over $t_{H(i)\setminus\{i\}}$ given $s_{H(i)}$, hence a function of $s_{H(i)}$ only. In strategy space the game is a network game with actions s_i of Tb bits and the same sets $H(i)$; apply Theorem 3. □

Remark 9 (Bit budget). Exhaustive search costs 2^{Nb}, Theorem 3 costs $2^{O(kbN/\sqrt{\log N})}$; a sufficient condition for an asymptotic improvement is $k \le c\sqrt{\log N}$ for an absolute constant c. The constant is not known: Claude [6] note that their procedure is existential when the constants of the pieces are not explicit. The cost in fact scales with the number of occurrences $\sum_i |H(i)|$ rather than with Nk.

5 Hardness with guaranteed existence

5.1 A parsimonious reduction

Let ϕ be a 3-CNF on $x_1, \dots, x_n$ with clauses $C_1, \dots, C_m$, each with one to three literals. The game $\Gamma(\phi)$ has binary actions and the following players; all payoffs lie in $\{0, 1\}$.

Player	Payoff equals 1 iff	Depends on
free $y_{v,0}$ ($v \in [n]$)	never (constant 0)	—
copy $y_{v,r}$, $r = 1, \dots, \mathrm{occ}(v)$	$y_{v,r} = y_{v,r-1}$	1 other
OR-checkers for $C = \ell_1 \vee \ell_2 \vee \ell_3$	$c_C^1 = \ell_1 \vee \ell_2$; $c_C = c_C^1 \vee \ell_3$	2 others each
p_C	$p_C = 0$ if $c_C = 1$; $p_C = q_C$ if $c_C = 0$	2 others
q_C	$q_C \neq p_C$	1 other

Literals are read from copies, one copy per occurrence; a clause with one or two literals uses one checker. The game has $N \leq n + 7m$ players and is built in linear time; every payoff depends on at most two other players ($k = 3$), which is the smallest neighbourhood for which existence is NP-hard [12].

Lemma 10. $|\mathrm{PSNE}(\Gamma(\phi))|$ *equals the number of satisfying assignments of* ϕ.

Proof. In a pure equilibrium every copy and every checker has a unique best response, so all copies of v equal $y_{v,0}$, encoding an assignment x, and $c_C = C(x)$. If $c_C = 0$, the pair (p_C, q_C) plays matching pennies and has no pure equilibrium; hence every clause is satisfied. Conversely, each satisfying x extends to exactly one equilibrium: free players are indifferent, copies and checkers are forced, and for $c_C = 1$ the pair is forced to $p_C = 0$, $q_C = 1$. A variable with no occurrence contributes a factor 2 on both sides; repeated literals receive separate copies; tautological clauses yield $c_C = 1$. □

Existence and counting hardness at these parameters are known [12]; we use Lemma 10 as a building block.

5.2 A switch construction

Given ϕ, add a variable z and set

$$\phi' = \bigwedge_C (\neg z \vee C) \ \wedge \ \bigwedge_{i=1}^{n} (z \vee \neg x_i).$$

Its solutions are $(z = 0, x = 0)$ and $(z = 1, x)$ for every x satisfying ϕ; hence $\#\phi' = 1 + \#\phi$. To keep neighbourhoods of size two, the clause $\neg z \vee C$ is checked by c_C as above followed by a checker d_C with unique best response $d_C = \neg z' \vee c_C$, where z' is a copy of z; the pennies pair is attached to d_C. Clauses $z \vee \neg x_i$ use one checker. Call the result $\Gamma'(\phi)$. By the argument of Lemma 10, $|\mathrm{PSNE}(\Gamma'(\phi))| = 1 + \#\phi$; in particular a pure equilibrium always exists.

Let W^* be the number of non-free players of $\Gamma'(\phi)$. In every pure equilibrium each non-free player earns 1, so every equilibrium has welfare W^*, the largest welfare any profile can have.

Theorem 11. *For network games with two actions per player, payoffs in* $\{0, 1, 2, 3\}$ *and each payoff depending on at most two other players:*

(a) deciding uniqueness of the pure equilibrium is coNP-hard, even when one is guaranteed to exist;

(b) deciding $\mathrm{PoS} = 1$ *is NP-hard and deciding* $\mathrm{PoA} = 1$ *is coNP-hard, even when an equilibrium is guaranteed to exist; hence computing* PoA *and* PoS *exactly is hard;*

(c) for a local objective, deciding whether the optimistic selection value is at least 1 *is NP-complete, and deciding whether the pessimistic selection value is at least* 1 *is coNP-complete;*

(d) for a leader with two commitments, each of which changes the followers' payoff tables or the leader's objective, deciding whether a given commitment is pessimistic-optimal is coNP-hard.

Proof. (a) The equilibrium of $\Gamma'(\phi)$ is unique iff ϕ is unsatisfiable.

(b) For PoS, add $2 \cdot \mathbf{1}[y_{z,0} = 1]$ to the payoff of the copy $y_{z,1}$; this term does not depend on that player's action, so best responses are unchanged. If ϕ is satisfiable, an equilibrium with $z = 1$ has welfare $W^* + 2 = \mathrm{OPT}$. If not, every equilibrium has $z = 0$ and welfare W^*. Consider instead the profile with $y_{z,0} = 1$, every other copy of z equal to 0, $x = 0$ (all copies of each x_i equal to 0), and every checker, every d_C and every pennies pair at its best response given these values. Each clause $\neg z \vee C$ is satisfied because its copy z' is 0, so $d_C = 1$ and the pair earns 2; each clause $z \vee \neg x_i$ is satisfied because $x_i = 0$. Only $y_{z,1}$ fails its rule, so this profile has welfare $W^* - 1 + 2 > W^*$, and $\mathrm{PoS} > 1$. For PoA, add $\mathbf{1}[y_{z,0} = 0]$ instead: if ϕ is unsatisfiable the unique equilibrium has welfare $W^* + 1 = \mathrm{OPT}$; otherwise an equilibrium with $z = 1$ has welfare $W^* < \mathrm{OPT}$. Welfare is positive in all equilibria, so the ratios are defined.

(c) Take $f = \mathbf{1}[y_{z,0} = 1]$: its maximum over equilibria is 1 iff ϕ is satisfiable. Take $f = \mathbf{1}[y_{z,0} = 0]$: its minimum over equilibria is 1 iff ϕ is unsatisfiable. Membership follows by exhibiting an equilibrium.

(d) Let $L = \{0, 1\}$. Under $\ell = 0$ the followers play $\Gamma'(\phi)$ and the leader's objective is $2 \cdot \mathbf{1}[y_{z,0} = 0]$. Under $\ell = 1$ the payoff table of the free player $y_{z,0}$ is changed to $\mathbf{1}[y_{z,0} = 0]$, which forces $z = 0$ in every equilibrium (such an equilibrium exists), and the leader's objective is $\mathbf{1}[y_{z,0} = 0]$. The pessimistic value of $\ell = 1$ is 1; that of $\ell = 0$ is 2 if ϕ is unsatisfiable and 0 otherwise. So $\ell = 0$ is pessimistic-optimal iff ϕ is unsatisfiable. □

Remark 12 (Verification and prior work). Lemma 10 was checked by exhaustive enumeration on 73 random formulas. The construction $\Gamma'(\phi)$ with two-input checkers, and statements (a)–(d), were checked by exact variable elimination (cross-validated against exhaustive enumeration on small games) on 124 random formulas with up to five variables and six clauses of width up to three, (97 satisfiable, 27 unsatisfiable), giving games with up to 78 players. Without the promise of existence, uniqueness is US-hard, since the reduction is parsimonious. For two-player games in normal form, Conitzer and Sandholm [10] show that deciding uniqueness of a Nash equilibrium and finding equilibria with high welfare are NP-hard; statements (a)–(d) are pure-strategy counterparts for succinct games with guaranteed existence. We have not found them for graphical games in the literature, but we do not claim that individual parts are new.

6 Tightness under the Stearns–Hunt hypothesis

Theorem 13. *Assume SH. For every $\varepsilon > 0$, none of the following problems for network games with two actions and payoffs depending on at most two other players can be decided in time $O(2^{N^{1-\varepsilon}})$: existence of a pure equilibrium, and the problems of Theorem 11(a)–(d). Under #SH the same holds for counting pure equilibria.*

Proof. The reductions of Section 5 run in polynomial time and map a 3-CNF of size $s = \Theta(n + m)$ to a game with $N = O(s)$ players and description size $O(s)$. Apply the transfer principle of MIT Hardness Group et al. [18] with $\gamma = 1$; deterministic time bounds are closed under complement, which covers the coNP-hard problems; for counting, use their counting analogue and the parsimony of Lemma 10. □

Conditional on Fact 2, Theorem 3 (with $k = 3$, $b = 1$) gives algorithms in time $2^{O(N/\sqrt{\log N})}$, while Theorem 13 rules out $2^{N^{1-\varepsilon}}$ for every fixed ε. In this sense the exponent is $N^{1-o(1)}$; the

gap between $N/\sqrt{\log N}$ and the SH lower bound is an $N^{o(1)}$ factor in the exponent and is not characterised. Under ETH the same reductions would have ruled out $2^{o(N)}$; conditional on the engine, that statement is false for these games.

7 Illustration: capacity procurement with supplier choice

Buyers $i \in [N]$ and suppliers $j \in [M]$ with integer capacities κ_j form a bipartite eligibility graph: buyer i may order from a set $E(i)$ of at most d suppliers, and each supplier is eligible for at most Δ buyers. Buyer i chooses $a_i = (j, q)$ with $j \in E(i)$ and q from a finite set $Q \ni 0$ of order levels ($q = 0$ means no order; all actions with $q = 0$ are identified). Let L_j be the total order at supplier j. Under proportional rationing $x_i = q_i$ if $L_j \le \kappa_j$ and $x_i = q_i\kappa_j/L_j$ otherwise, and payoffs take the form $u_i = r_{ij}\min(x_i, \delta_i) - w_j x_i$, where δ_i is buyer i's need, in the spirit of Cachon and Lariviere [4].

The algorithm applies. Buyer i's payoff, and each deviation payoff, depends only on buyers who may use a supplier in $E(i)$, so $|H(i)| \le d\Delta$, and actions take $b = \lceil \log(d|Q|) \rceil$ bits. Conditional on Fact 2, Corollary 4 counts pure equilibria in time $2^{O(d\Delta\, b\, N/\sqrt{\log N})}$, in exact rational arithmetic. With payoffs scaled to integers bounded by $\mathrm{poly}(N)$—for instance with an integer rationing rule, or with Δ and $|Q|$ constant so that a constant common denominator suffices—Corollaries 5–7 give the exact price of anarchy and stability and the optimistic and pessimistic values of a regulator or platform choosing among a polynomial list of rules (for example priority classes or reservation levels that change table entries).

Two easy cases.

Proposition 14. *(i) If every buyer's supplier is fixed ($d = 1$), the game splits into independent single-supplier games with at most Δ players each. (ii) If all buyers share one supplier and order levels are integers in $\{0, \dots, Q_{\max}\}$, then the number of pure equilibria, the best and worst equilibrium welfare, and* OPT *can be computed in time* $\mathrm{poly}(N, Q_{\max})$.

Proof. (i) is immediate. (ii) Payoffs depend on (q_i, L) with $L = \sum_j q_j \le NQ_{\max}$. Fix a candidate total L. Buyer i's order q_i is a best response iff $u_i(q_i, L) \ge u_i(q_i', L - q_i + q_i')$ for all q_i', a condition on (q_i, L) only; let $F_i(L)$ be the set of such q_i. Pure equilibria with total L are the vectors $q \in \prod_i F_i(L)$ with $\sum_i q_i = L$. Their number, and the maximum and minimum of $\sum_i u_i(q_i, L)$ over them, follow from a dynamic program over partial sums in time $O(N^2 Q_{\max}^2)$; the same program without the restriction to $F_i(L)$ gives OPT for this L. Summing (or optimising) over L gives the claim. □

Conjecture 15. *Deciding whether the capacity procurement game above has a pure Nash equilibrium is NP-hard for some constant $d, \Delta, |Q|$.*

Some care is needed: if all orders have the same size and payoffs decrease in load, the game is a singleton congestion game with player-specific payoffs and always has a pure equilibrium [17]. With order-level choice we have found, by computer search, two-buyer instances without pure equilibria and small configurations that behave like matching pennies and like a NOR gate when their inputs are held fixed; composing them requires controlling how downstream load feeds back on upstream buyers, which we leave open.

8 Limits of the method

1. **Dense interaction.** The cost is exponential in the number of shared bits. Games in which payoffs depend on aggregates of many players (outside special cases such as Proposition 14(ii)), or two-player games with large strategy sets, exceed the bit budget.

2. **Large payoffs.** Optimisation over equilibria is reduced to counting via generating functions, which requires integer payoffs bounded by a polynomial.

3. **Nested quantifiers.** The engine sums with signs; it does not evaluate max–min expressions. A leader with exponentially many commitments, for example one choosing a capacity for every supplier, gives $\exists \ell \, \forall a \in \mathrm{PSNE}(\Gamma_\ell)$ statements that the method does not handle.

4. **Mixed equilibria.** Approximate mixed equilibria are governed by PPAD-type hypotheses untouched by the refutation of ETH [20].

5. **Practice and verification.** The algorithms beat exhaustive search only for astronomically large inputs, and they rest on claims that had not been independently refereed at the time of writing; so does the statement that the exponent is $N^{1-o(1)}$. The hardness results (Section 5) and the SH lower bounds (Theorem 13) do not depend on those claims.

9 Conclusion

Conditional on the claimed refutation of ETH, exact equilibrium analysis of sparse network games moves from exponential to subexponential time, for a whole family of questions at once, through one generating function. The same reductions that once gave $2^{\Omega(N)}$ lower bounds give, under SH, lower bounds of $2^{N^{1-\varepsilon}}$. The boundary of the method is the number of bits players share. Whether natural operations models such as capacity procurement with supplier choice are hard in the same sense is open (Conjecture 15).

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