Frank–Wolfe Theorems for Mixed Integer Quadratic Optimization

Alberto Del Pia* Brandon Sommerfeld†

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Abstract

The Frank–Wolfe theorem states that a quadratic function bounded from below on a polyhedron attains its infimum. We study when this remains true in the presence of integer variables, for sets defined by quadratic inequalities. Our main result is a complete classification for sets defined by convex quadratic inequalities with rational coefficients and a rational affine hull, a condition that holds for all full-dimensional sets. One integer variable never destroys attainment: if every rational quadratic objective that is bounded from below attains its infimum over the set, even after adding rational linear constraints, the same holds with one integer variable. In dimension at most 2, integer variables never destroy attainment. In dimension at least 3 and with two or more integer variables, attainment holds for every such objective if and only if the set has bounded curvature, that is, it is bounded except along directions in which all constraints are linear. Without convexity, these results remain valid, except that bounded curvature is no longer necessary. Moreover, integer variables can destroy attainment only if there are at least two of them and at least one continuous variable. The result for one integer variable follows from a lifting theorem, which also extends known results for cubic objectives and convex polynomial constraints. Finally, for convex constraints and rational data, attainment holds when the objective is bounded from above, convex, or quasi-convex over a full-dimensional convex set containing the feasible region.

Keywords: Frank–Wolfe theorem; Mixed integer quadratic optimization; Existence of optimal solutions; Quadratically constrained quadratic programming.

1 Introduction

A classical theorem of Frank and Wolfe [12] states that a quadratic function that is bounded from below on a polyhedron attains its infimum. Results of this kind are called *Frank–Wolfe type theorems*, and many of them are known for continuous problems, see Section 1.3. In this paper, we ask when they remain true in the presence of integer variables.

Setting. We consider mixed integer quadratically constrained quadratic programs of the form

$$\begin{aligned} &\inf\ f_0(x) \\ &\text{s.t. } x \in S \cap P \cap \Lambda. \end{aligned} \tag{1}$$

Here

$$S := \{x \in \mathbb{R}^n : f_i(x) \leq 0, \quad i = 1, 2, \ldots, m\}, \tag{2}$$

*Department of Industrial and Systems Engineering & Wisconsin Institute for Discovery, University of Wisconsin–Madison, Madison, WI, USA. delpia@wisc.edu

†Department of Computer Sciences, University of Wisconsin–Madison, Madison, WI, USA. bdsommerfeld@wisc.edu

where, for each $j = 0, 1, \ldots, m$,

$$f_j(x) = x^\mathsf{T} Q_j x + q_j^\mathsf{T} x + c_j,$$

with a symmetric matrix $Q_j \in \mathbb{R}^{n\times n}$, $q_j \in \mathbb{R}^n$, and $c_j \in \mathbb{R}$. Moreover, $P \subseteq \mathbb{R}^n$ is a polyhedron, and Λ is a *mixed integer lattice*, that is, a set of the form

$$\Lambda = \tau\left(\mathbb{Z}^p \times \mathbb{R}^{n-p}\right),$$

where $\tau(x) = Mx + \bar{x}$ for an invertible $M \in \mathbb{R}^{n\times n}$ and $\bar{x} \in \mathbb{R}^n$. We say that Λ has p *integer components*. For $p = 0$, we obtain the continuous case $\Lambda = \mathbb{R}^n$. We say that (1) *attains its infimum* if there exists $x \in S \cap P \cap \Lambda$ such that $f_0(x) \le f_0(y)$ for every $y \in S \cap P \cap \Lambda$. In particular, the infimum is then finite. To avoid trivial cases, we assume throughout the paper that $S \cap P \cap \Lambda$ is nonempty.

An instance of (1) has *rational data* if all f_j have *rational coefficients*, that is, all entries of Q_j, q_j, and c_j are rational, P is *rational*, that is, defined by finitely many linear inequalities with rational coefficients, and Λ is *rational*, that is, M and $\bar{x}$ can be chosen rational. In this introduction, we always assume rational data.

The Frank–Wolfe property. Whether an optimization problem has an optimal solution is one of the most basic questions in optimization. Algorithms, optimality conditions, and complexity results often assume that the infimum is attained. For mixed integer quadratic optimization, this leads to a fundamental question: for which sets S is the infimum in (1) attained for every f_0, P, and Λ for which it is finite? For continuous variables, that is, if only $\Lambda = \mathbb{R}^n$ is allowed, the Frank–Wolfe theorem shows that $S = \mathbb{R}^n$ has this property, and Frank–Wolfe type theorems give other such sets. This question leads to the following definition.

Definition 1.1. A set $S \subseteq \mathbb{R}^n$ has the *Frank–Wolfe property* FW_p, or simply is FW_p, if, for every rational mixed integer lattice Λ with at most p integer components, every rational polyhedron P, and every quadratic function f_0 with rational coefficients that is bounded from below on $S \cap P \cap \Lambda \neq \emptyset$, problem (1) attains its infimum.

The intersection with P makes the property hereditary: if S is FW_p, then so is $S \cap P$ for every rational polyhedron P. Moreover, FW_p implies FW_{p-1}, and FW_0 only concerns continuous variables. In this language, the Frank–Wolfe theorem shows that $\mathbb{R}^n$ is FW_0, and hence so is every rational polyhedron.

1.1 Main result

The following notion turns out to characterize the Frank–Wolfe property for convex constraints.

Definition 1.2. Let S be defined as in (2), and let $N := \bigcap_{i=1}^m \ker Q_i$. We say that S has *bounded curvature* if $S \subseteq B + N$ for a bounded set B.

In words, S is bounded except along the directions in N, along which all constraints are linear. Equivalently, the orthogonal projection of S onto $N^\perp = \sum_{i=1}^m \operatorname{range} Q_i$ is bounded. For example, polyhedra and compact sets have bounded curvature, while the paraboloid $\{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \le x_3\}$ does not.

Our main result is the following complete classification.

Theorem 1.3. *Let S be given by* (2)*, where all f_i are convex with rational coefficients, and assume that the affine hull of S is rational. Then*

(i) S is FW_1 if and only if S is FW_0.

(ii) If the affine hull of S has dimension at most 2, then S is FW_n if and only if S is FW_0.

(iii) If the affine hull of S has dimension at least 3, then for every $2 \le p \le n$, the set S is FW_p if and only if S has bounded curvature.

In words, one integer variable never destroys attainment, and in dimension at most 2, no number of integer variables does. This follows from Theorem 3.1, which lifts attainability results from the purely continuous setting to the one integer setting. In dimension at least 3 and with two or more integer variables, attainment for all quadratic objectives is equivalent to bounded curvature. The assumption on the affine hull holds for full-dimensional sets, and for sets defined by one convex quadratic inequality and linear inequalities [9, Theorem 1.2].

The threshold 3 in (iii) is sharp. The set $\{x \in \mathbb{R}^2 : x_1^2 \le x_2\}$ is FW_0 by [16, Theorem 2], and hence FW_2 by (ii), but it does not have bounded curvature. For a full-dimensional set in $\mathbb{R}^3$ as in Theorem 1.3, the theorem leaves three possibilities: the set is FW_3, it is FW_1 but not FW_2, or it is not FW_0. All three occur. Sets with bounded curvature are FW_3. The set $\{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \le x_3\}$ is FW_1 by [16, Theorem 2] and (i), but it is not FW_2 by Example 4.5. Finally, the set $\{x \in \mathbb{R}^3 : x_2^2 \le x_1,\ x_3^2 \le x_1 + 1\}$ is not even FW_0 [6, Section 3].

1.2 Extensions and consequences

We now discuss which parts of Theorem 1.3 extend to nonconvex constraints, what our results imply about the role of integer variables, how conditions on the objective can replace bounded curvature, and what happens without rational data. We conclude by relating our results to those of Luo and Zhang [16].

Nonconvex constraints. Without convexity, parts (i) and (ii) of Theorem 1.3 remain valid, see Corollaries 3.2 and 4.11. In part (iii), bounded curvature remains sufficient for S to be FW_n, see Corollary 4.3, but it is no longer necessary. Remark 4.4 gives a nonconvex set that is FW_n but does not have bounded curvature.

The role of integer variables. Let S be given by (2), convex or not, and assume that S is FW_0, so that attainment holds with continuous variables only. Then integer variables can destroy attainment only if there are at least two of them, by Corollary 3.2, and at least one continuous variable, since attainment always holds when all variables are integer, see Corollary 4.10. Moreover, this can only happen if S does not have bounded curvature, by Corollary 4.3.

Conditions on the objective. Theorem 1.3 concerns only the set S, and it quantifies over all mixed integer lattices and all quadratic objectives. Under additional assumptions on the objective, the infimum is attained even when S does not have bounded curvature, as the following result shows.

Theorem 1.4. *Consider an instance of* (1) *with rational data, where all f_i are convex and f_0 is bounded from above and from below on $S \cap P \cap \Lambda$. Then f_0 attains its infimum on $S \cap P \cap \Lambda$.*

We prove Theorem 1.4 as Theorem 5.1 in Section 5. Together with its consequences for convex and quasi-convex objectives, Theorem 1.4 shows that, for convex constraints, the objective in any nonattainment example is nonconvex and unbounded from above, see Remark 5.3.

Irrational data. So far we assumed rational data. With one integer variable, the lifting theorem in Section 3 does not need rational data. With two integer variables, rational data are needed, even for polyhedra, see Appendix C, where Table 1 summarizes both settings.

Relation to Luo and Zhang. Our results extend to the mixed integer setting the results of Luo and Zhang [16] on attainment in continuous quadratically constrained quadratic programming. First, they showed that a set S defined by a single convex quadratic inequality and linear inequalities is FW_0 [16, Theorem 2]. By Corollary 3.2, such a set is always FW_1, and if S has dimension at most 2 and a rational affine hull, then it will be FW_n by Corollary 4.11. For S which has dimension at least 3 and a rational affine hull, S is FW_p for any $2 \le p \le n$ if and only if S has bounded curvature by Corollary 4.7. Additionally, for a single convex quadratic inequality $x^\mathsf{T} Q x + q^\mathsf{T} x + c \le 0$ with $Q \neq 0$ and $n \ge 3$, Corollary 4.8 shows that S is FW_n if and only if $q \in \operatorname{range} Q$.

Second, Luo and Zhang showed that the infimum is attained when all constraints are convex and f_0 is quasi-convex over the polyhedron defined by the linear constraints [16, Theorem 3]. Corollary 5.2 shows that, for any number of integer variables, it suffices that f_0 is convex or quasi-convex over some full-dimensional convex set containing S. For at most one integer variable, Corollary E.5 only requires that f_0 is quasi-convex over S. Luo and Zhang also showed that nonconvex constraints can define sets that are not FW_0, even when the feasible region is convex [16, Example 3].

1.3 Related literature

Frank–Wolfe type theorems for continuous problems have been studied extensively. Alternative proofs and extensions of the Frank–Wolfe theorem are given in [8, 11, 26]. Attainment is also known for cubic objectives with linear constraints [1, 14], for convex polynomial objectives and constraints [5, 6], and for other classes of objectives and constraints [10, 21, 22]. Ozdaglar and Tseng [25] and Bertsekas and Tseng [7] gave unified existence results based on recession and asymptotic directions, and Bertsekas and Tseng mention extensions with integer variables as a topic for future research [7, Section 4].

Other works characterize the feasible sets on which Frank–Wolfe type theorems hold. Kummer [15] proved such a theorem for sets of the form $K + D$, where K is compact and D is a polyhedral cone. Martínez-Legaz et al. [18, Theorem 12.2] proved the converse: if every quadratic function that is bounded from below on $K + D$ attains its infimum there, for a closed convex cone D, then D is polyhedral. This is a continuous analog of Theorem 4.6. They also characterize the convex sets on which every quadratic objective that is quasi-convex on the set and bounded from below attains its infimum [17, Theorem 3.1]. Corollary 5.2 is a result of this type in the mixed integer setting.

In the mixed integer setting, Meyer [19] proved attainment for linear objectives and linear constraints with rational coefficients, and Bank and Hansel [3] for quadratic objectives with linear constraints. Bank and Mandel [4] extended this result to sets of the form $K + V$, where K is compact and V is generated by finitely many integer vectors. Corollary 4.3 follows from their result, and the converse in Theorem 1.3(iii) is new. For integer programs with convex or reverse convex constraints, Obuchowska [23, 24] studied boundedness and the existence of optimal solutions.

Organization of the paper. Section 2 introduces our notation. Section 3 lifts attainment results from continuous variables to one integer variable, and proves part (i) of Theorem 1.3. Section 4 considers two or more integer variables and proves the rest of Theorem 1.3. Section 5 considers conditions on the objective and proves Theorem 1.4. The appendix contains the omitted proofs, further consequences of the lifting theorem, and the results on irrational data.

2 Preliminaries

Throughout the paper, f_j, Q_j, q_j, and c_j always refer to an instance of (1). Results that are more general than (1) use other notation, typically g_j. Since a linear inequality is a quadratic inequality with $Q_i = 0$,

in results about a single instance of (1) we include the inequalities defining P among those defining S, and write $S \cap \Lambda$ for the feasible region.

We say that a direction $d \in \mathbb{R}^n$ is a *recession direction* of an inequality $f(x) \leq 0$ where $f : \mathbb{R}^n \to \mathbb{R}$, if $f(x + td) \leq 0$ for every x with $f(x) \leq 0$ and $t \geq 0$. Similarly, d is a *recession direction* of a set $X \subseteq \mathbb{R}^n$ if

$$\forall x \in X, \forall t \geq 0, x + td \in X.$$

We denote by $\mathrm{rec}(X)$ the set of recession directions of X. The *dimension* of a set is the dimension of its affine hull. We denote by e^i the i-th standard unit vector, whose dimension is clear from the context. An affine subspace of $\mathbb{R}^n$ is *rational* if it can be defined by linear equations with rational coefficients, or equivalently, if it is of the form $\{x_0 + Wz : z \in \mathbb{R}^k\}$ with $x_0 \in \mathbb{Q}^n$ and $W \in \mathbb{Q}^{n \times k}$.

Throughout the paper, when we refer to a quadratic function $g : \mathbb{R}^n \to \mathbb{R}$, the canonical form is $g(x) = x^\mathsf{T} Q x + q^\mathsf{T} x + c$, where $Q \in \mathbb{R}^{n \times n}$ is a symmetric matrix, $q \in \mathbb{R}^n$, and $c \in \mathbb{R}$. The function g is convex if and only if Q is positive semidefinite [27, Theorem 4.5].

Remark 2.1. All examples in this paper can be extended by any number of additional integer or continuous variables that appear neither in the constraints nor in the objective. Hence they generalize to higher dimensions.

Our proofs use four elementary lemmas on recession directions and rational approximation. We state and prove them in Appendix A.

3 One integer variable

In this section, we show that attainability in the continuous setting can be lifted to the single integer setting. Specifically, we consider problems of the form

$$\begin{aligned} &\inf\ g_0(x) \\ &\text{s.t. } g_i(x) \leq 0, \quad i = 1, 2, \ldots, m, \\ &\quad x \in \Lambda, \end{aligned} \tag{3}$$

where for $j = 0, 1, \ldots, m$, $g_j : \mathbb{R}^n \to \mathbb{R}$ is a general function, and $\Lambda = M\left(\mathbb{Z} \times \mathbb{R}^{n-1}\right) + \bar{x}$ for an invertible $M \in \mathbb{R}^{n \times n}$ and $\bar{x} \in \mathbb{R}^n$. Thus Λ is a mixed integer lattice with one integer component, and in this section it need not be rational. Let $a^* := M^{-\mathsf{T}} e^1$, so that $\Lambda = \{x \in \mathbb{R}^n : (a^*)^\mathsf{T}(x - \bar{x}) \in \mathbb{Z}\}$.

We also utilize the related continuous problem with additional constraints, given by

$$\begin{aligned} &\inf\ g_0(x) \\ &\text{s.t. } g_i(x) \leq 0, \quad i = 1, 2, \ldots, m, \\ &\quad L \leq a^\mathsf{T} x \leq U \\ &\quad x \in \mathbb{R}^n, \end{aligned} \tag{4}$$

obtained by relaxing the mixed integer lattice in (3) and adding bounds $L, U \in \mathbb{R} \cup \{-\infty, \infty\}$ on $a^\mathsf{T} x$ for some $a \in \mathbb{R}^n$, but keeping the same functions g_j for $j = 0, 1, \ldots, m$.

Attaining the infimum is defined as for (1).

Theorem 3.1. *Consider a problem of the form* (3) *with finite infimum c, and assume that the set*

$$\{x \in \mathbb{R}^n : g_0(x) \leq c, g_i(x) \leq 0, \quad i = 1, 2, \ldots, m\}$$

has finitely many connected components. Assume that for $a = a^*$ and every choice of $L, U \in \left((a^*)^\mathsf{T}\bar{x} + \mathbb{Z}\right) \cup \{-\infty, \infty\}$ for which (4) *has a finite infimum, this infimum is attained. Then* (3) *attains its infimum.*

The idea of the proof is as follows. Consider a sequence of feasible points whose objective values converge to the infimum. If their integer coordinates are bounded, then only finitely many slices with a fixed integer coordinate are involved, and each slice attains its infimum by assumption. Otherwise, the sublevel set of the continuous relaxation at the infimum has finitely many connected components, and each of them lies strictly between two consecutive values of the integer coordinate. Beyond the last of them, a relaxation with one additional linear constraint has the same infimum but does not attain it.

We prove Theorem 3.1 in Appendix B.

We now prove part (i) of Theorem 1.3, for every set given by (2).

Corollary 3.2. *For any S defined as in* (2)*, S is FW_0 if and only if S is FW_1.*

Proof. By definition, every FW_1 set is FW_0, so we only need to prove the converse.

Let P be a rational polyhedron. Consider the set

$$S_c := \{x \in \mathbb{R}^n : f_0(x) \le c, f_i(x) \le 0, \quad i = 1, 2, \dots, m\}.$$

Milnor [20, Theorem 3] shows that any set defined by polynomial inequalities has a finite sum of Betti numbers, and this also applies to $S_c \cap P$ since P is given by linear inequalities. Since the 0th Betti number is the number of connected components, this shows that for any c the number of connected components of $S_c \cap P$ is finite.

Since S is FW_0, so is $S \cap P$. Mixed integer lattices with no integer components are covered by the definition of FW_0. Consider a rational mixed integer lattice Λ with one integer component, and an instance of (3) with $\{x \in \mathbb{R}^n : g_i(x) \le 0,\ i = 1, 2, \dots, m\} = S \cap P$ and an objective g_0 that is quadratic with rational coefficients. Since Λ is rational, a^* and $(a^*)^\mathsf{T}\bar{x}$ are rational. Hence each relaxation (4) in Theorem 3.1 only adds rational linear inequalities, and it attains its infimum when this is finite. By Theorem 3.1, problem (3) attains its infimum. Hence S is FW_1. □

In the same way, Theorem 3.1 lifts other attainment results from continuous variables to one integer variable. In Appendix B, we extend the results of Luo and Zhang [16, Theorem 2] for one convex quadratic constraint, of Klatte [14] for cubic objectives, and of Belousov and Klatte [6, Theorem 3] for convex polynomials. These extensions do not require rational data or a rational mixed integer lattice. In contrast, rational data are needed with two integer variables, see Appendix C.

4 Two or more integer variables

In this section, we consider mixed integer lattices with at least two integer components. We first show that bounded curvature implies the property FW_n, for every set with rational coefficients. We then show that bounded curvature is necessary for convex sets of dimension at least 3 with a rational affine hull. Finally, we treat lattices with n integer components and sets of dimension at most 2.

Lemma 4.1. *Let S be defined as in* (2)*, and assume that S is nonempty. Let $N := \bigcap_{i=1}^m \ker Q_i$ and $R := \{w \in N : q_i^\mathsf{T} w \le 0,\ i = 1, 2, \dots, m\}$.*

(1) If S has bounded curvature, then $S = K + R$ for a compact set K.

(2) If all f_i are convex, then $R = \mathrm{rec}(S)$, and the following are equivalent:

(a) S has bounded curvature.

(b) $S = K + V$, where K is a compact set and V is a polyhedral cone.

(c) $S = K + \mathrm{rec}(S)$, where K is a compact set.

We prove Lemma 4.1 in Appendix D.

The following result is due to Bank and Mandel [4, Section 6.2, Theorem 2 with Lemma 2(ii)].

Theorem 4.2. *Let $S = C + V$ where C is compact and V is a cone generated by finitely many integer vectors. Let $f(x) = x^\mathsf{T} Dx + q^\mathsf{T} x + c$ for $D \in \mathbb{Q}^{n\times n}$, $q \in \mathbb{R}^n$, $c \in \mathbb{R}$. If f is bounded from below on $S \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p}) \neq \emptyset$, then it attains its infimum there.*

This gives the following corollary, which proves one direction of Theorem 1.3(iii), without convexity.

Corollary 4.3. *Let S be given by* (2)*, where all f_i have rational coefficients, and assume that S has bounded curvature. Then S is FW_n.*

Proof. Let $\Lambda = \tau(\mathbb{Z}^p \times \mathbb{R}^{n-p})$ be a rational mixed integer lattice and let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rational polyhedron. After the change of variables $x = \tau(y)$, the lattice becomes $\mathbb{Z}^p \times \mathbb{R}^{n-p}$, and the transformed set is still defined by quadratic inequalities with rational coefficients. It still has bounded curvature, since $\tau^{-1}(B+N) = M^{-1}(B-\bar{x}) + M^{-1}N$, and $M^{-1}N$ is the intersection of the kernels of the transformed quadratic forms. Hence we may assume that $\Lambda = \mathbb{Z}^p \times \mathbb{R}^{n-p}$. The set $S \cap P$ is defined by the constraints of S and the linear inequalities of P, so it also has bounded curvature. By Lemma 4.1(1), we can write $S \cap P = C + R$ for a compact set C, where $R = \{w \in N : q_i^\mathsf{T} w \leq 0,\ i = 1, 2, \ldots, m,\ Aw \leq 0\}$. This cone is defined by rational linear equations and inequalities, so it is generated by finitely many rational vectors, which can be scaled to be integer. Thus Theorem 4.2 applies to $S \cap P$ for every quadratic objective with rational coefficients, and S is FW_n. □

Remark 4.4. The converse of Corollary 4.3 does not hold without convexity. For example, let $S := \{x \in \mathbb{R}^n : 1 - x_1^2 \leq 0\}$. Its orthogonal projection onto $N^\perp = \mathrm{span}(e^1)$ is unbounded, so S does not have bounded curvature. However, for every rational polyhedron P, the set $S \cap P$ is the union of the rational polyhedra $P \cap \{x : x_1 \geq 1\}$ and $P \cap \{x : x_1 \leq -1\}$. By [28, Theorem 8.5], each of them is the sum of a polytope and a cone generated by finitely many integer vectors. As in the proof of Corollary 4.3, Theorem 4.2 applies to each of them. If f_0 is bounded from below on $S \cap P \cap \Lambda$, it attains its infimum on the intersection of each piece with Λ when this is nonempty, and the smaller of these minima is the minimum over $S \cap P \cap \Lambda$. Hence S is FW_n.

We now show the converse, for sets of dimension at least 3 with a rational affine hull, given by convex quadratics with rational coefficients, and for $2 \leq p \leq n-1$.

The following example shows that a single convex quadratic constraint can prevent attainment. It illustrates the construction in the proof of Theorem 4.6.

Example 4.5. For every $n, p \in \mathbb{N}$ such that $2 \leq p \leq n-1$, there exists an instance of (1) with a rational mixed integer lattice Λ with p integer components, a nonconvex f_0, and a single convex quadratic constraint f_1, both with rational coefficients, such that the infimum is finite but not attained.

Proof. Consider the instance of (1) as

$$\begin{gathered} \inf x_3 - x_2^2 \\ \text{s.t. } x_1^2 + x_2^2 - x_3 \leq 0 \\ x \in \{(t, 2\lambda_1 + t, 2\lambda_2 + 1) : t \in \mathbb{R}, \lambda_1, \lambda_2 \in \mathbb{Z}\}. \end{gathered}$$

On the feasible region, the objective satisfies $x_3 - x_2^2 \geq x_1^2 \geq 0$, with equality if and only if $x_1 = 0$ and $x_3 = x_2^2$. The mixed integer lattice is a union of lines that do not intersect this parabola.

For some feasible $x = (t, 2\lambda_1 + t, 2\lambda_2 + 1)$ to have an objective value of 0, it would require $t = 0$, but then we would also need $(2\lambda_1)^2 = 2\lambda_2 + 1$, which is impossible as $4\lambda_1^2$ is even while $2\lambda_2 + 1$ is odd.

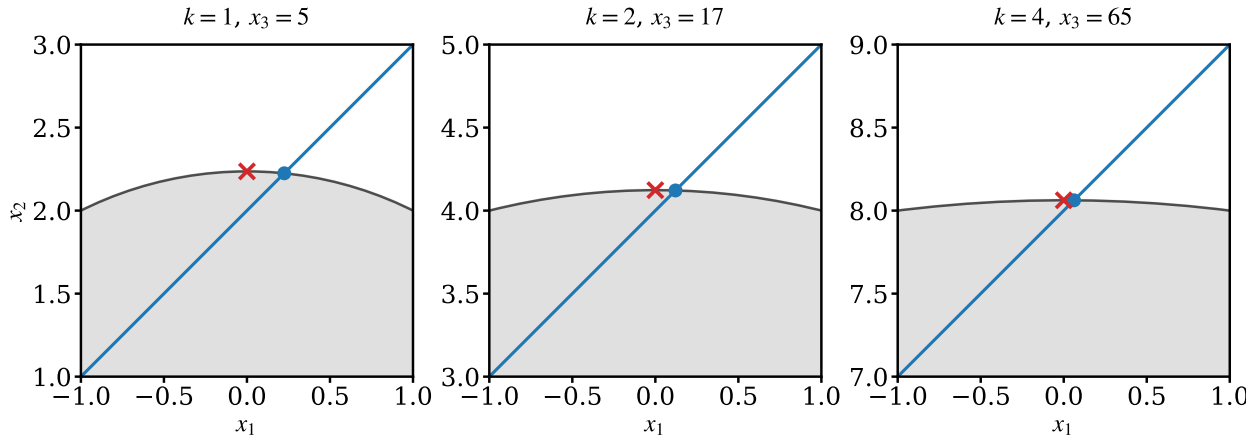


Figure 1: Example 4.5 in the planes $x_3 = 4k^2 + 1$. The shaded region is the slice of S, and the line is the slice of the lattice line with $x_2 = x_1 + 2k$. The dot is the point x^k, with objective value t_k^2. The cross is the point with $x_1 = 0$ and $x_3 = x_2^2$, which has objective value 0 but is not in the lattice.

Now fix λ_1 and λ_2. The values of t for which the constraint holds with equality are the roots of a quadratic equation, and one of them is

$$t = \frac{-2\lambda_1 + \sqrt{2 - 4\lambda_1^2 + 4\lambda_2}}{2}. \tag{5}$$

If this is a real number, the corresponding point has objective value t^2. We show that t can be made arbitrarily small.

For $k = 1, 2, \ldots$, let $\lambda_{1,k} := k$ and $\lambda_{2,k} := 2k^2$, so that $2 - 4\lambda_{1,k}^2 + 4\lambda_{2,k} = 4k^2 + 2$, and let t_k be given by (5). Then $x^k := (t_k, 2\lambda_{1,k} + t_k, 2\lambda_{2,k} + 1)$ is feasible with objective value t_k^2, and $t_k = \sqrt{k^2 + 1/2} - k = \frac{1}{2(\sqrt{k^2+1/2}+k)}$. Hence $\lim_{k\to\infty} t_k = 0$, so the infimum is 0, and it is not attained. Figure 1 shows these points.

By Remark 2.1, adding integer or continuous variables gives an example for all n and p with $2 \le p \le n - 1$. □

Theorem 4.6. *Let S be given by* (2)*, where all f_i are convex with rational coefficients. Assume that S has dimension at least 3, has a rational affine hull, and does not have bounded curvature. Then for every p with $2 \le p \le n - 1$, there exist a quadratic function f_0 with rational coefficients, a rational mixed integer lattice Λ with p integer components, and a rational polyhedron H such that f_0 has a finite infimum over $S \cap H \cap \Lambda$, but does not attain it.*

Proof (sketch). After a rational change of variables, we may assume that S is full-dimensional. Since S does not have bounded curvature, there are points $x^k \in S$ whose projections u^k onto $N^\perp$ satisfy $\|u^k\| \to \infty$. We write $x^k = u^k + w^k$ with $w^k \in N$ of norm $O(\|u^k\|^2)$. The constraints that curve in the limit direction e of $u^k/\|u^k\|$ force $\|w^k\|$ to grow quadratically in $\|u^k\|$, and the limit direction of $w^k/\|w^k\|$ is a recession direction of S along which these constraints decrease. We replace these two limit directions by rational directions $\bar{e}$ and d with the same properties.

Together with a third rational direction v, the directions $\bar{e}$ and d span a three-dimensional affine subspace H through a rational interior point of S, with coordinates (s, t, y). For all large s and all y in a small interval, points in $S \cap H$ satisfy $t \ge \psi(s, y)$, where ψ is a quadratic function with rational coefficients. The objective $f_0 = t - \psi(s, y) + (y - \theta)^2$, for a rational θ, is nonnegative on $S \cap H$, and it vanishes only on the curve where

$y = \theta$ and $t = \psi(s, \theta)$. The lattice in H consists of the points with $s = k + \lambda\sigma$, $t \in \mathbb{Z} + \frac{1}{2D}$, and $y = \sigma$, where $k \in \mathbb{Z}$, $\sigma \in \mathbb{R}$, $\lambda \in \{1, 2\}$, and D is a common denominator of the values $\psi(k + \lambda\theta, \theta)$. It does not meet the curve, since these values lie in $\frac{1}{D}\mathbb{Z}$. The value of ψ changes at rate proportional to k, so a change of σ of order $\frac{1}{k}$ reaches a point with $t = \psi(s, y)$ and $t \in \mathbb{Z} + \frac{1}{2D}$. These lattice points have objective values of order $\frac{1}{k^2}$, so the infimum is 0 and it is not attained. Finally, we add $p - 2$ integer and $n - p - 1$ continuous directions to obtain a lattice in $\mathbb{R}^n$ with p integer components. □

Example 4.5 follows a similar construction, with $\bar{e} = e^2$, $d = e^3$, $v = e^1$, $\psi(s, y) = s^2 + y^2$, and $\theta = 0$. There, the objective is $x_3 - x_2^2 = t - \psi(s, y) + y^2$, and the lattice consists of the points with $s - y \in 2\mathbb{Z}$ and $t \in 2\mathbb{Z} + 1$. The full proof of Theorem 4.6 is in Appendix D.

Together, these results give part (iii) of Theorem 1.3.

Corollary 4.7. *Let S be given by* (2)*, where all f_i are convex with rational coefficients. Assume that S has dimension at least 3 and a rational affine hull, and let $2 \le p \le n$. Then S is FW_p if and only if S has bounded curvature.*

Proof. By Corollary 4.3, if S has bounded curvature, then it is FW_n, and hence FW_p. Conversely, if S does not have bounded curvature, Theorem 4.6 gives a counterexample with 2 integer components, which exists since $n \ge 3$. Hence S is not FW_p for any $p \ge 2$. □

Every full-dimensional set with $n \ge 3$ satisfies the assumptions of Corollary 4.7. So does every set of dimension at least 3 defined by a single convex quadratic inequality and linear inequalities with rational coefficients, since its affine hull is rational by [9, Theorem 1.2]. For a single convex quadratic inequality, we obtain the following characterization, which we prove in Appendix D.

Corollary 4.8. *Let $n \ge 3$, and let $S := \{x \in \mathbb{R}^n : x^\mathsf{T} Q x + q^\mathsf{T} x + c \le 0\}$, where Q is positive semidefinite and nonzero, and Q, q, and c are rational. Then S is FW_2 if and only if $q \in \operatorname{range} Q$. In this case, S is FW_n.*

To complete the picture, we also have the following elementary result [4, Section 7.3, Remark 4].

Theorem 4.9. *Let $X \subseteq \mathbb{Z}^n$ be nonempty. If f is a polynomial with rational coefficients which is bounded from below on X, then f attains its infimum on X.*

This gives the following corollary for mixed integer lattices with n integer components.

Corollary 4.10. *Let S be given by* (2) *with $n \ge 1$, let Λ be a rational mixed integer lattice with n integer components, let P be a rational polyhedron, and let f_0 be a quadratic function with rational coefficients that is bounded from below on $S \cap P \cap \Lambda \ne \emptyset$. Then f_0 attains its infimum on $S \cap P \cap \Lambda$. Consequently, S is FW_n if and only if S is FW_{n-1}, and if $n \le 2$, then S is FW_n if and only if S is FW_0.*

Proof. Write $\Lambda = \tau(\mathbb{Z}^n)$. Then $f_0 \circ \tau$ is a polynomial with rational coefficients that is bounded from below on the nonempty set $\tau^{-1}(S \cap P \cap \Lambda) \subseteq \mathbb{Z}^n$. By Theorem 4.9, it attains its infimum there, so f_0 attains its infimum on $S \cap P \cap \Lambda$. By definition, every FW_n set is FW_{n-1}, and the converse follows from the first statement. For $n \le 2$, the last statement then follows from Corollary 3.2 in Section 3. □

Finally, we consider sets of dimension at most 2. This proves part (ii) of Theorem 1.3, without convexity.

Corollary 4.11. *Let S be given by* (2)*, and assume that the affine hull of S is rational and has dimension at most 2. Then S is FW_n if and only if S is FW_0.*

Proof (sketch). By [9, Theorem 1.1], a rational affine map identifies the affine hull of S, intersected with Λ, with a mixed integer lattice in $\mathbb{R}^k$, where $k \le 2$. The corresponding set in $\mathbb{R}^k$ is FW_0, and hence FW_k by Corollary 4.10. The full proof is in Appendix D. □

5 Conditions on the objective

In this section, we show that the infimum of (1) is attained under assumptions on the objective that are not covered by Theorem 1.3. The main result is Theorem 5.1: if the data are rational, the constraints are convex, and f_0 is bounded from above and from below on $S \cap \Lambda$, then the infimum is attained.

The idea of the proof of the following theorem is as follows. If $S \cap \Lambda$ is bounded, then the infimum is attained. Otherwise, there is a rational recession direction d of the constraints. Since f_0 is bounded on $S \cap \Lambda$, it is constant along d on $S \cap \Lambda$. After adding two linear constraints, it is constant along d on the continuous relaxation, and we can project out d and use induction on n.

Theorem 5.1. *Consider an instance of* (1) *with rational data. If all constraints f_i $(i = 1, 2, \ldots, m)$ are convex quadratic functions, and $f_0(x)$ is bounded from above on $S \cap \Lambda$, then* (1) *attains its infimum when it is finite.*

We prove Theorem 5.1 in Appendix E, using three lemmas.

Corollary 5.2. *Consider an instance of* (1) *with rational data, where all constraints f_i $(i = 1, \ldots, m)$ are convex. Assume that f_0 is convex, or that f_0 is quasi-convex over a full-dimensional convex set $C \supseteq S \cap \Lambda$. Then* (1) *attains its infimum (when it is finite).*

Proof. First, assume that f_0 is convex, that is, Q_0 is positive semidefinite. Let $r \in \mathbb{Q}$ with $r > \inf_{y \in S \cap \Lambda} f_0(y)$, and add the constraint $f_0(x) - r \leq 0$. This constraint is convex and has rational coefficients. A point attains the infimum of the original problem if and only if it attains the infimum of the new problem. In the new problem, f_0 is bounded from above on the feasible region, so the infimum is attained by Theorem 5.1.

Next, assume that Q_0 is not positive semidefinite. Then f_0 is quasi-convex but not convex over a full-dimensional convex set $C \supseteq S \cap \Lambda$. By [2, Theorem 6.3], applied to $-f_0$, the function f_0 is bounded from above on C (see also [16, Lemma 7]), and thus on $S \cap \Lambda$. The infimum is attained by Theorem 5.1. □

If S is full-dimensional and f_0 is quasi-convex over S, we can take $C = S$. If the polyhedron defined by the linear constraints is full-dimensional and f_0 is quasi-convex over it, as in [16, Theorem 3], we can take C to be this polyhedron.

For at most one integer variable, Corollary E.5 in Appendix E removes the rationality assumption of Corollary 5.2, and it only requires that the objective is quasi-convex over S.

Remark 5.3. Theorem 5.1 and Corollary 5.2 restrict the objectives in Theorem 4.6. There, the constraints that define $S \cap H$ are convex with rational coefficients. Hence the objective f_0 is not convex, it is not quasi-convex over any full-dimensional convex set containing $S \cap H$, and it is unbounded from above on $S \cap H \cap \Lambda$. In particular, in this setting every linear objective with rational coefficients attains its infimum when it is finite. The objective $x_3 - x_2^2$ of Example 4.5 has these properties.

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A Auxiliary lemmas

We collect some lemmas that are used in several proofs. The first lemma describes the recession directions of a set defined by convex quadratic inequalities.

Lemma A.1. *Let* $X := \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, 2, \ldots, m\}$*, where each* $g_i(x) = x^\mathsf{T} Q_i x + q_i^\mathsf{T} x + c_i$ *is convex, and assume that* X *is nonempty. Then* $\operatorname{rec}(X) = \{d \in \mathbb{R}^n : Q_i d = 0,\ q_i^\mathsf{T} d \leq 0,\ i = 1, 2, \ldots, m\}$*. Moreover,* $d \in \operatorname{rec}(X)$ *if and only if* d *is a recession direction of each inequality* $g_i(x) \leq 0$*.*

Proof. Let d satisfy $Q_i d = 0$ and $q_i^\mathsf{T} d \leq 0$ for every i. Then $g_i(x + td) = g_i(x) + tq_i^\mathsf{T} d \leq g_i(x)$ for every $x \in \mathbb{R}^n$, $t \geq 0$, and i. Hence d is a recession direction of each inequality, and of X. Conversely, let $x \in X$, and let d be a recession direction of X or of each inequality. In both cases, for every i and every $t \geq 0$ we have

$$0 \geq g_i(x + td) = g_i(x) + t\left(2x^\mathsf{T} Q_i d + q_i^\mathsf{T} d\right) + t^2 d^\mathsf{T} Q_i d.$$

Dividing by t^2 and letting $t \to \infty$ gives $d^\mathsf{T} Q_i d \leq 0$. Since Q_i is positive semidefinite, this implies $Q_i d = 0$. Then $g_i(x) + tq_i^\mathsf{T} d \leq 0$ for every $t \geq 0$, so $q_i^\mathsf{T} d \leq 0$. □

The next lemma replaces a nonzero vector in a rational polyhedron by a nearby nonzero rational vector.

Lemma A.2. *Let* $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ *with* $A \in \mathbb{Q}^{m \times n}$ *and* $b \in \mathbb{Q}^m$*, and let* $\bar{x} \in P \setminus \{0\}$*. Then for every* $\epsilon > 0$ *there exists* $x \in (P \setminus \{0\}) \cap \mathbb{Q}^n$ *with* $\|x - \bar{x}\| < \epsilon$*.*

Proof. Let B be a box defined by rational inequalities, with side length less than $\frac{\epsilon}{\sqrt{n}}$, such that $\bar{x} \in B$ and $0 \notin B$. Then $P \cap B$ is a nonempty polyhedron defined by rational inequalities, since it contains $\bar{x}$. By [28, Theorem 10.1], it contains a rational point x. This point is nonzero since $0 \notin B$, and $\|x - \bar{x}\| < \epsilon$ since both points lie in B. □

The following lemma is implicit in the proof of [16, Theorem 1].

Lemma A.3. *Consider a convex quadratic inequality*

$$f(x) = x^\mathsf{T} Q x + q^\mathsf{T} x + c \leq 0. \tag{6}$$

Let $\left(x^k\right)$ *be a sequence of solutions of* (6) *with* $\lim_{k\to\infty} \|x^k\| = \infty$ *such that the limit* $d := \lim_{k\to\infty} \frac{x^k}{\|x^k\|}$ *exists. Then* d *is a recession direction of* (6)*.*

Proof. Taking the inequality for x^k and dividing through by $\|x^k\|^2$ yields

$$\left(\frac{x^k}{\|x^k\|}\right)^\mathsf{T} Q \left(\frac{x^k}{\|x^k\|}\right) + q^\mathsf{T} \frac{x^k}{\|x^k\|^2} + \frac{c}{\|x^k\|^2} \leq 0.$$

Taking the lim sup as k tends to infinity gives $d^\mathsf{T} Q d \leq 0$, but since Q is positive semidefinite this means $Qd = 0$.

If we instead only divide by $\|x^k\|$ we get

$$\frac{1}{\|x^k\|}(x^k)^\mathsf{T} Q x^k + q^\mathsf{T} \frac{x^k}{\|x^k\|} + \frac{c}{\|x^k\|} \leq 0.$$

Then we must have $q^\mathsf{T} d \leq 0$, because the first term is always nonnegative and the last term tends to 0, so $q^\mathsf{T} d > 0$ would imply $f(x^k) > 0$ for large enough values of k.

Then Lemma A.1 gives that d is a recession direction for (6). □

The following lemma is used several times. It shows that moving along a rational direction by integer multiples of a suitable step stays in a mixed integer lattice.

Lemma A.4. *Let $d \in \mathbb{Q}^n$, let $p \in \{0, 1, \dots, n\}$, and let $\Lambda = \tau\,(\mathbb{Z}^p \times \mathbb{R}^{n-p})$ for an invertible rational affine map τ. Then there exists a rational $t > 0$ such that $x + ktd \in \Lambda$ for every $x \in \Lambda$ and every $k \in \mathbb{Z}$.*

Proof. After the change of variables $x = \tau(y)$, we may assume that $\Lambda = \mathbb{Z}^p \times \mathbb{R}^{n-p}$, since d is replaced by the rational vector $M^{-1}d$. Write $d_i = \frac{a_i}{b_i}$ with $a_i \in \mathbb{Z}$ and $b_i \in \mathbb{Z}_{>0}$. Then $t := \prod_{i=1}^{p} b_i$, where the empty product is 1, ensures that the first p components of $x + ktd$ are integer. □

B One integer variable

This appendix contains the proof of Theorem 3.1 and further consequences of it.

Proof (of Theorem 3.1). After the change of variables $x = My + \bar{x}$, we may assume that $\Lambda = \mathbb{Z} \times \mathbb{R}^{n-1}$ and that the relaxations in the assumption are those with bounds $L, U \in \mathbb{Z} \cup \{-\infty, \infty\}$ on x_1. Indeed, $(a^*)^\mathsf{T} x = y_1 + (a^*)^\mathsf{T}\bar{x}$. To refer to the feasible regions of the problems, we define

$$K := \{x \in \mathbb{R}^n : g_i(x) \le 0, \quad i = 1, 2, \dots, m\},$$

$$K_{L,U} := \{x \in \mathbb{R}^n : L \le x_1 \le U, g_i(x) \le 0, \quad i = 1, 2, \dots, m\},$$

and $T := \{x \in \mathbb{R}^n : g_0(x) \le c\}$. By assumption, $T \cap K$ has finitely many connected components.

Now, assume for contradiction that (3) does not attain its infimum. Then no $x \in K \cap (\mathbb{Z} \times \mathbb{R}^{n-1})$ satisfies $g_0(x) \le c$, and there is a sequence (x^k) in $K \cap (\mathbb{Z} \times \mathbb{R}^{n-1})$ with $\lim_{k\to\infty} g_0(x^k) = c$.

First, assume that the sequence (x_1^k) is bounded. Since its elements are integers, it takes finitely many values j. For each such j, the relaxation (4) with $L = U = j$ has feasible region $K_{j,j} \subseteq K \cap (\mathbb{Z} \times \mathbb{R}^{n-1})$, so its infimum γ_j is finite and at least c. By assumption, γ_j is attained, so $\gamma_j > c$. Then $g_0(x^k) \ge \min_j \gamma_j > c$ for every k, a contradiction.

Hence (x_1^k) is unbounded. Passing to a subsequence, we may assume that x_1^k tends to ∞, as the case where it tends to $-\infty$ is symmetric. Every $x \in T \cap K$ satisfies $x_1 \notin \mathbb{Z}$, since otherwise x would attain the infimum. Let C be a connected component of $T \cap K$. Its projection onto the first coordinate is connected, and thus an interval in $\mathbb{R}$ that contains no integer. Hence there is $n_C \in \mathbb{Z}$ such that $n_C < x_1 < n_C + 1$ for every $x \in C$. Since $T \cap K$ has finitely many connected components, we can take $B := \max_C n_C + 1$, or any $B \in \mathbb{Z}$ if $T \cap K$ is empty. Then $T \cap K_{B,\infty}$ is empty. Hence every feasible point of the relaxation (4) with $L = B$ and $U = \infty$ has objective value larger than c. On the other hand, $x^k \in K_{B,\infty}$ for all k large enough, and $\lim_{k\to\infty} g_0(x^k) = c$. Hence the infimum of this relaxation is c. By assumption, it is attained, so there is $x \in K_{B,\infty}$ with $g_0(x) = c$. This contradicts that $T \cap K_{B,\infty}$ is empty. □

Theorem 3.1 also lifts other known attainment results from continuous variables to one integer variable. This applies to every class of problems defined by polynomials that remains in the class when linear inequalities are added. The sets in Theorem 3.1 are then semialgebraic, so they have finitely many connected components. If the continuous result allows linear inequalities with real coefficients, the mixed integer lattice Λ need not be rational. The proofs are the same as the proof of Corollary 3.2, so we omit them.

For example, we can generalize [16, Theorem 2].

Corollary B.1. *Consider a problem of the form* (3)*, where g_0 is a quadratic function, all constraint functions g_i $(i = 1, 2, \dots, m)$ are convex quadratic functions, and at most one of them is not affine. Then* (3) *attains its infimum (when it is finite).*

The following lifts the cubic programming attainability from Klatte [14] to the mixed integer setting.

Corollary B.2. *Consider a problem of the form*

$$\begin{aligned} &\inf\ g_0(x) \\ &\text{s.t. } Ax \le b \\ &\quad x \in \Lambda, \end{aligned} \tag{7}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, g_0 is a polynomial of degree at most 3, and Λ is as in (3)*.*
Then (7) *attains its infimum (when it is finite).*

We can also lift the result of Belousov and Klatte for convex polynomials [6, Theorem 3].

Corollary B.3. *Consider a problem of the form*

$$\begin{aligned} &\inf\ g_0(x) \\ &\text{s.t. } g_i(x) \le 0, \quad i = 1, 2, \dots, m \\ &\quad x \in \Lambda, \end{aligned} \tag{8}$$

where $g_j : \mathbb{R}^n \to \mathbb{R}$ for $j = 0, 1, \dots, m$ is a convex polynomial, and Λ is as in (3)*.*
Then (8) *attains its infimum (when it is finite).*

Corollaries B.1, B.2, and B.3 do not require rational data or a rational mixed integer lattice. In contrast, the next section shows that rational data are needed with two integer variables.

C Irrational data

In this section, we show that rational data are needed in general once there are two integer variables, even for linear constraints.

We first recall two results of Meyer. Theorem C.1 is the example in [19, Section 2], and Theorem C.2 follows from [19, Theorem 2.1].

Theorem C.1. *There exists an instance of* (1) *with $n = p = 2$ and Λ rational, where all constraints and the objective function are affine, but some constraint has irrational coefficients, such that the infimum is finite but not attained.*

Theorem C.2. *Consider an instance of* (1) *where Λ is rational, all functions f_j $(j = 0, \dots, m)$ are affine, and all constraints f_i $(i = 1, \dots, m)$ have rational coefficients. Then the infimum is attained whenever it is finite.*

Hence, when all functions are affine, rationality of the constraints alone ensures that the infimum is attained, and this assumption cannot be dropped in general. The next example shows that rational constraints are not enough for a quadratic objective: an objective with irrational coefficients may not attain its infimum.

Example C.3. There exist instances of (1) where Λ is rational, all constraints f_i $(i = 1, \dots, m)$ are affine with rational coefficients, but the objective f_0 is a quadratic function with irrational coefficients, such that the infimum is finite but not attained.

Proof. Consider

$$\begin{aligned} &\inf\ (x_1 - x_2\sqrt{2})^2 \\ &\text{s.t. } x_2 \ge 1 \\ &\quad x \in \mathbb{Z}^2. \end{aligned}$$

Table 1: When can integer variables destroy attainment for a set S given by (2) that is FW_0, convex or not? Here p is the number of integer components of the mixed integer lattice.

	Rational data	Irrational data
$p \le 1$	never (Corollary 3.2)	never, if the continuous result allows real data (Section 3)
$2 \le p \le n-1$	only if S does not have bounded curvature (Corollary 4.3), and exactly then if S is convex with a rational affine hull of dimension at least 3 (Theorem 1.3(iii))	possible, even for polyhedra (Theorem C.1 and Remark 2.1)
$p = n \ge 2$	never (Corollary 4.10)	possible, even for polyhedra (Theorem C.1 and Remark 2.1)

An objective value of 0 cannot be attained for any $x \in \mathbb{Z}^2$, because that would require $\frac{x_1}{x_2} = \sqrt{2}$. For $k = 1, 2, \ldots$, define positive integers a_k, b_k by $a_k + b_k\sqrt{2} := (1+\sqrt{2})^k$, and let $x^k := (a_k, b_k)$. Then $a_k - b_k\sqrt{2} = (1-\sqrt{2})^k$, so the objective value of x^k is $(\sqrt{2}-1)^{2k}$, which tends to 0. Since $b_k \ge 1$, the points x^k are feasible. Hence the infimum is 0, and it is not attained. □

Both examples have two integer variables. With one integer variable, the infimum is attained in both settings, even if the mixed integer lattice is not rational. This follows from Corollary B.2, since the objectives are polynomials of degree at most 2 and the constraints are linear.

Table 1 summarizes when integer variables can destroy attainment for a set that is FW_0, with and without rational data.

D Two or more integer variables

This appendix contains the proofs omitted from Section 4.

Proof (of Lemma 4.1). Let $U := N^\perp$, and let π_U and π_N denote the orthogonal projections onto U and N. For every $x \in \mathbb{R}^n$ and $w \in N$, we have $f_i(x+w) = f_i(x) + q_i^\mathsf{T} w$, since $Q_i w = 0$. In particular, $x + r \in S$ for every $x \in S$ and $r \in R$.

For (1), write each $x \in S$ as $x = u + w$, where $u = \pi_U(x)$ and $w = \pi_N(x)$. Since S has bounded curvature, $\pi_U(S)$ is contained in a bounded set B. For each $u \in \pi_U(S)$, the set of $w \in N$ such that $u + w \in S$ is

$$F_u := \{w \in N : q_i^\mathsf{T} w \le -f_i(u), \quad i = 1, 2, \ldots, m\}.$$

This polyhedron is nonempty, and its recession cone is R. By [28, Theorem 8.5], we can write $F_u = \mathrm{conv}(Z_u) + R$, where $Z_u \subseteq F_u$ is a finite set that contains an arbitrary point of each minimal face of F_u.

Each minimal face of F_u is of the form $\{w \in N : q_i^\mathsf{T} w = -f_i(u),\ i \in I'\}$ for some $I' \subseteq \{1, 2, \ldots, m\}$, and we choose its point of minimum norm. This point is $G_{I'} b_{I'}(u)$, where $b_{I'}(u) := (-f_i(u))_{i \in I'}$ and $G_{I'}$ is a matrix that depends only on I' and on the constraints. Since $u \in B$ is bounded, $f_i(u)$ is also bounded. Hence each element of Z_u lies in a bounded set B' that is independent of u. We can take B' convex, so that $\mathrm{conv}(Z_u) \subseteq B'$. Then we can write $w = \bar{w} + r$, where $\bar{w} \in B' \cap F_u$ and $r \in R$. Since u and $\bar{w}$ are bounded, we have $\|u + \bar{w}\| \le \rho$ for a constant ρ, and $u + \bar{w} \in S$ since $\bar{w} \in F_u$. Hence $S \subseteq K + R$, where $K := S \cap \{x \in \mathbb{R}^n : \|x\| \le \rho\}$ is compact. The reverse inclusion $K + R \subseteq S$ holds by the first paragraph of the proof.

For (2), Lemma A.1 gives $\mathrm{rec}(S) = R$. Then (a) $\Rightarrow$ (c) follows from (1), and (c) $\Rightarrow$ (b) holds since R is a polyhedral cone. For (b) $\Rightarrow$ (a), we have $V \subseteq \mathrm{rec}(S) = R \subseteq N$. Hence $\pi_U(S) \subseteq \pi_U(K)$, which is bounded. □

The next lemma characterizes bounded curvature for a set defined by a single convex quadratic inequality.

Lemma D.1. *Let S be given as in* (2) *for a single convex quadratic constraint f_1 with $Q_1 \neq 0$. Then S has bounded curvature if and only if $q_1 \in \operatorname{range} Q_1$. Moreover, if $q_1 \notin \operatorname{range} Q_1$ then S is full-dimensional.*

Proof. Let $N = \ker Q_1$ and $U = N^\perp$, and let π_U denote orthogonal projection onto U. If $q_1 \in \operatorname{range} Q_1$, let $\bar{x} \in \mathbb{R}^n$ satisfy $q_1 = -2Q_1\bar{x}$. Then for any $x \in \mathbb{R}^n$ we can write

$$x^\mathsf{T} Q_1 x + q_1^\mathsf{T} x + c_1 = (x - \bar{x})^\mathsf{T} Q_1 (x - \bar{x}) + c',$$

for some constant c'. Then $(x - \bar{x})^\mathsf{T} Q_1 (x - \bar{x}) = u^\mathsf{T} Q_1 u$, where $u = \pi_U(x - \bar{x})$. But Q_1 is positive definite on U, so the set defined by $u^\mathsf{T} Q_1 u + c' \leq 0$ is an ellipsoid, which is bounded. Thus, $\pi_U(S)$ is bounded, so S has bounded curvature.

Now suppose $q_1 \notin \operatorname{range} Q_1$, so we can write $q_1 = q_U + q_N$ for $q_U \in U, q_N \in N \setminus \{0\}$. Then we have $Q_1 q_N = 0$ and $q_1^\mathsf{T} q_N = \|q_N\|^2$, so for $u \in U, t \in \mathbb{R}$ we can write

$$f_1(u - tq_N) = u^\mathsf{T} Q_1 u + q_U^\mathsf{T} u + c_1 - t\|q_N\|^2.$$

For large enough t, this is negative, so every $u \in U$ belongs to $\pi_U(S)$. Since $Q_1 \neq 0$, U is unbounded, showing S does not have bounded curvature. Moreover, this also shows there are $x \in S$ with $f_1(x) < 0$, which is an interior point since f_1 is continuous. □

Proof (of Corollary 4.8). If $q \in \operatorname{range} Q$, then S has bounded curvature by Lemma D.1, so S is FW_n by Corollary 4.3. Otherwise, S is full-dimensional and does not have bounded curvature by Lemma D.1, so S is not FW_2 by Corollary 4.7. □

Proof (of Theorem 4.6). We first reduce to the case where S is full-dimensional. Let $\ell \geq 3$ be the dimension of S. Since the affine hull of S is rational, it is $\{x_0 + Wz : z \in \mathbb{R}^\ell\}$ for some $x_0 \in \mathbb{Q}^n$ and $W \in \mathbb{Q}^{n \times \ell}$ with full column rank. The set $S_W := \{z \in \mathbb{R}^\ell : f_i(x_0 + Wz) \leq 0,\ i = 1, 2, \ldots, m\}$ is full-dimensional and defined by convex quadratic inequalities with rational coefficients, and $S = x_0 + WS_W$. If $S_W = K_W + \operatorname{rec}(S_W)$ for a compact set K_W, then $S = (x_0 + WK_W) + \operatorname{rec}(S)$, since $\operatorname{rec}(S) = W \operatorname{rec}(S_W)$. Hence, by Lemma 4.1, S_W does not have bounded curvature. Let p_W be an integer with $\max\{2, p - n + \ell\} \leq p_W \leq \min\{\ell - 1, p\}$, which exists since $\ell \geq 3$ and $2 \leq p \leq n - 1$. Assume that the theorem holds for S_W and p_W, with objective f_0^W, rational mixed integer lattice $\Lambda_W \subseteq \mathbb{R}^\ell$, and rational polyhedron H_W. Let $g^1, \ldots, g^{n-\ell} \in \mathbb{Q}^n$ be such that the columns of W and these vectors form a basis of $\mathbb{R}^n$, and define

$$\Lambda := \left\{ x_0 + Wz + \sum_{j=1}^{n-\ell} \lambda_j g^j : z \in \Lambda_W,\ \lambda \in \mathbb{Z}^{p-p_W} \times \mathbb{R}^{n-\ell-p+p_W} \right\}.$$

This is a rational mixed integer lattice with p integer components, and $\Lambda \cap \{x_0 + Wz : z \in \mathbb{R}^\ell\} = x_0 + W\Lambda_W$, since the vectors g^j and the columns of W are linearly independent. Let $W^+ := (W^\mathsf{T} W)^{-1} W^\mathsf{T}$, let $H := x_0 + WH_W$, which is a rational polyhedron, and let $f_0(x) := f_0^W(W^+(x - x_0))$, which is a quadratic function with rational coefficients. Since $W^+ W$ is the identity, the map $z \mapsto x_0 + Wz$ is a bijection from $S_W \cap H_W \cap \Lambda_W$ to $S \cap H \cap \Lambda$ that preserves objective values. Hence f_0 has a finite infimum over $S \cap H \cap \Lambda$ that is not attained. Therefore, from now on we assume that S is full-dimensional, so that $n \geq 3$.

Let $N = \bigcap_{i=1}^m \ker Q_i$ and $U = N^\perp$. Our goal is to find directions in the set S with specific properties, which we can use to build our counterexample.

Let π_U denote the orthogonal projection map onto U, and π_N likewise for N. Then S not having bounded curvature implies there is a sequence (x^k) such that $\lim_{k\to\infty} \|\pi_U(x^k)\| = \infty$. Let $u^k = \pi_U(x^k)$. Then if we consider

$$F_k := \{w \in N : q_i^\mathsf{T} w \leq -f_i(u^k) \quad i = 1, 2, \ldots, m\},$$

we have $u^k + w \in S$ iff $w \in F_k$, and F_k is nonempty since it contains $\pi_N(x^k)$. First, we note that for each constraint, there is a constant L_i such that for every $u \in \mathbb{R}^n$, $|f_i(u)| \leq L_i(1 + \|u\|^2)$. Combined with Hoffman's error bound [13] applied to 0, we get that there is a constant L independent of k such that there is $w^k \in F_k$ with $\|w^k\| \leq L(1 + \|u^k\|^2)$. Then we replace x^k with $u^k + w^k$, and work with this sequence instead.

Now, pass to a subsequence such that the limit $\lim_{k\to\infty} \left(\frac{u^k}{\|u^k\|}\right) = e \in U$ exists. We show below that $w^k \neq 0$ for all sufficiently large k. Passing to a further subsequence, we may assume that the limit $\lim_{k\to\infty} \left(\frac{w^k}{\|w^k\|}\right) = \tilde{d}$ also exists. We show that e and $\tilde{d}$ have some useful properties. First, since $e \in U \setminus \{0\}$, we have $e \notin N$, so the set

$$I := \{i \in \{1, 2, \ldots, m\} : e^\mathsf{T} Q_i e > 0\},$$

is nonempty. For $\tilde{d}$, let $i \in I$ and let $a_i = e^\mathsf{T} Q_i e > 0$. Then we have that $\lim_{k\to\infty} \frac{f_i(u^k)}{\|u^k\|^2} = a_i$, so for large enough k we have $f_i(u^k) \geq \frac{1}{2} a_i \|u^k\|^2$. Then we have $q_i^\mathsf{T} w^k \leq -f_i(u^k) \leq -\frac{1}{2} a_i \|u^k\|^2$, so $q_i \neq 0$ and $\|w^k\| \geq \frac{a_i}{2\|q_i\|} \|u^k\|^2$, meaning $\|w^k\|$ is $\Theta(\|u^k\|^2)$.

This implies that $\lim_{k\to\infty} \frac{x^k}{\|x^k\|} = \tilde{d}$, so by Lemma A.3 applied to each constraint, $\tilde{d}$ is a recession direction of S. We further have that

$$q_i^\mathsf{T} \tilde{d} = \lim_{k\to\infty} \frac{q_i^\mathsf{T} w^k}{\|w^k\|} \leq \limsup_{k\to\infty} -\frac{1}{2} a_i \frac{\|u^k\|^2}{L(1 + \|u^k\|^2)} = -\frac{a_i}{2L} < 0.$$

Since $i \in I$ was arbitrary, we have $q_i^\mathsf{T} \tilde{d} < 0$ for each $i \in I$.

That is, for every constraint for which moving in the direction e causes a quadratic increase, moving in the direction $\tilde{d}$ causes those constraints to decrease. Our next step is to replace these directions with rational directions with the same properties. For $\tilde{d}$, since the recession cone of S is given by rational inequalities by Lemma A.1, we can apply Lemma A.2 to get a rational recession direction d of S, and by taking ϵ small enough we still have $q_i^\mathsf{T} d < 0$ for each $i \in I$.

Let $J := \{i \in \{1, 2, \ldots, m\} : q_i^\mathsf{T} d = 0\}$. These are the constraints where moving along d does not change the value, and note that it is disjoint from I. We want to replace e with a direction which is both rational, and does not behave badly on the constraints indexed by J. Define the cone

$$C := \left\{\rho = \sum_{i\in J} \mu_i q_i : \mu \geq 0, \pi_N(\rho) = 0\right\} \subseteq U.$$

Now we have that for each $x \in S$ the inequality $q_i^\mathsf{T} x \leq -c_i$ holds for each $i = 1, 2, \ldots, m$, because the quadratic term is always nonnegative. Then for any $\rho \in C$ and any k, we can write

$$\rho^\mathsf{T} x^k = \rho^\mathsf{T} u^k \leq -\sum_{i\in J} \mu_i c_i,$$

since $\rho^\mathsf{T} w^k = 0$ by definition of C. Dividing this inequality by $\|u^k\|$ and taking the limit as $k \to \infty$, we get $\rho^\mathsf{T} e \leq 0$, so $e \in C^\circ$, the polar cone of C.

This is the property we wanted e to have, and now we show how to find a rational direction close to e which keeps the desired properties. C can be expressed as the image of $\{\mu \geq 0 : \pi_N\left(\sum_{i\in J} \mu_i q_i\right) = 0\}$ under a rational linear map, so C is a rational polyhedral cone. This means it has a finite set of rational generators $\rho_1, \ldots, \rho_r$. Then we have

$$C^\circ = \{x \in \mathbb{R}^n : \rho_i^\mathsf{T} x \leq 0 \quad i = 1, 2, \ldots, r\},$$

which is a rational polyhedron. Also consider $Z = U \cap \bigcap_{i\notin I} \ker Q_i$, and we have $e \in Z$, and want to preserve this as well. Then we can apply Lemma A.2 to $C^\circ \cap Z$ to get a rational element $e' \in C^\circ \cap Z$, and by taking

ϵ small enough we still have $e'^{\mathsf{T}} Q_i e' > 0$ for $i \in I$, and the point of intersecting with Z is that we ensure we still have $Q_i e' = 0$ for $i \notin I$.

Now, we want to adjust e' to be well behaved on each constraint indexed by J, since those are the constraints where d does not cause a decrease. Towards this end, we want to find an adjustment vector δ such that $q_i^{\mathsf{T}}(e' + \delta) \leq 0$ for each $i \in J$. Let V_N be a matrix whose columns are a rational basis for N, so we can write the condition as finding a vector ω such that $q_i^{\mathsf{T}} V_N \omega \leq -q_i^{\mathsf{T}} e'$ for each $i \in J$. By Farkas' lemma, this system is infeasible iff there is $\mu \geq 0$ with

$$\sum_{i \in J} \mu_i V_N^{\mathsf{T}} q_i = 0, \sum_{i \in J} -\mu_i q_i^{\mathsf{T}} e' < 0.$$

But the first condition says that $\rho = \sum_{i \in J} \mu_i q_i$ has $\pi_N(\rho) = 0$, while the second says $\rho^{\mathsf{T}} e' > 0$, which contradicts that $e' \in C^\circ$, so the system is actually feasible, and has a rational solution $\bar{\omega}$, and we can then take $\delta = V_N \bar{\omega}$ as our adjustment vector, and define $\bar{e} = e' + \delta$.

Since $\delta \in N$, we still have $\bar{e}^{\mathsf{T}} Q_i \bar{e} > 0$ iff $i \in I$, and now $q_i^{\mathsf{T}} \bar{e} \leq 0$ for $i \in J$, while for d we have $q_i^{\mathsf{T}} d < 0$ for $i \notin J$, and $I \cap J = \emptyset$.

Now to actually construct our counterexample using these directions, fix $\bar{x} \in \operatorname{int} S \cap \mathbb{Q}^n$, which exists since S is full-dimensional, and let $v \in \mathbb{Q}^n \setminus \operatorname{span}(\bar{e}, d)$. Fix a rational ϵ such that $\bar{x} + yv \in S$ for all $|y| \leq \epsilon$. Let

$$H := \{x(s,t,y) = \bar{x} + s\bar{e} + td + yv : s,t,y \in \mathbb{R}\}.$$

Note that H is 3 dimensional: $\bar{e}$ and d are linearly independent, since $\bar{e}^{\mathsf{T}} Q_i \bar{e} > 0$ while $Q_i d = 0$ for each $i \in I$. Now we can express our constraints on H as

$$\begin{aligned} i \in I: &\qquad f_i(x(s,t,y)) = \alpha_i s^2 + \beta_i(y) s + f_i(\bar{x} + yv) + t q_i^{\mathsf{T}} d \\ i \notin I: &\qquad f_i(x(s,t,y)) = f_i(\bar{x} + yv) + s q_i^{\mathsf{T}} \bar{e} + t q_i^{\mathsf{T}} d \end{aligned}$$

Where $\alpha_i = \bar{e}^{\mathsf{T}} Q_i \bar{e}$ and $\beta_i(y) = 2(\bar{x} + yv)^{\mathsf{T}} Q_i \bar{e} + q_i^{\mathsf{T}} \bar{e}$. Then if we define for $i \in I$ the function

$$\psi_i(s,y) = -\frac{\alpha_i s^2 + \beta_i(y) s + f_i(\bar{x} + yv)}{q_i^{\mathsf{T}} d},$$

which is well defined since $q_i^{\mathsf{T}} d < 0$ for $i \in I$, we have $f_i(x(s,t,y)) \leq 0$ iff $t \geq \psi_i(s,y)$. Thus, we note

$$S \cap H \subseteq \left\{ x(s,t,y) : t \geq \max_{i \in I} \psi_i(s,y) \right\}.$$

Now, we order the functions ψ_i, $i \in I$. Let $\gamma_i := -\frac{1}{q_i^{\mathsf{T}} d} > 0$, so that

$$\psi_i(s,y) = \gamma_i \alpha_i s^2 + \gamma_i \beta_i(y) s + \gamma_i f_i(\bar{x} + yv).$$

For a fixed value of y, consider the triple $\gamma_i\ (\alpha_i, \beta_i(y), f_i(\bar{x} + yv))$, and order these triples lexicographically. For two distinct indices $i, j \in I$, the affine functions $\gamma_i \beta_i(y)$ and $\gamma_j \beta_j(y)$ are equal for at most one value of y, unless they are equal everywhere. Similarly, the quadratic functions $\gamma_i f_i(\bar{x} + yv)$ and $\gamma_j f_j(\bar{x} + yv)$ are equal for at most two values of y, unless they are equal everywhere. Thus, we can find an interval $[y_1, y_2] \subseteq (-\epsilon, \epsilon)$ with $y_1 < y_2$ that avoids these finitely many values. Then the lexicographic order of the triples is the same for all $y \in [y_1, y_2]$, where two indices whose triples are equal for every y are tied.

Let $i^* \in I$ be an index whose triple is largest in this order. We claim that there is s_0 such that $\psi_{i^*}(s,y) \geq \psi_i(s,y)$ for every $i \in I$, $s \geq s_0$, and $y \in [y_1, y_2]$. Indeed, $\psi_{i^*}(s,y) - \psi_i(s,y) = as^2 + b(y)s + c(y)$, where a is a constant, b is affine, and c is quadratic. If a, b, and c are identically zero on $[y_1, y_2]$, the claim

is clear. Otherwise, the first of them that is not identically zero is positive on $[y_1, y_2]$. Since it is continuous on the compact interval $[y_1, y_2]$, it is bounded away from 0, while the later coefficients are bounded. This gives s_0.

We show that for all sufficiently large s, every $y \in [y_1, y_2]$, and every $t \geq \psi_{i^*}(s, y)$, we have $x(s, t, y) \in S$. We do this by cases on the constraint.

- $i \in I$, then $f_i(x(s,t,y)) = q_i^\mathsf{T} d(t - \psi_i(s,y)) \leq q_i^\mathsf{T} d(\psi_{i^*}(s,y) - \psi_i(s,y))$ using that $q_i^\mathsf{T} d < 0$ for $i \in I$. By the claim, this is ≤ 0 for $s \geq s_0$, showing $x(s,t,y)$ satisfies this constraint.
- $i \in J$, then $f_i(x(s,t,y)) = f_i(\bar{x} + yv) + sq_i^\mathsf{T}\bar{e}$ since $q_i^\mathsf{T} d = 0$ for $i \in J$. Then $f_i(\bar{x} + yv) \leq 0$ since $|y| \leq \epsilon$, and since $q_i^\mathsf{T}\bar{e} \leq 0$ for $i \in J$, for $s \geq 0$ we have $x(s,t,y)$ feasible for this constraint.
- $i \in \{1, 2, \ldots, m\} \setminus (I \cup J)$. Then $f_i(x(s,t,y)) = f_i(\bar{x} + yv) + sq_i^\mathsf{T}\bar{e} + tq_i^\mathsf{T} d$ and $q_i^\mathsf{T} d < 0$. Since $\gamma_{i^*}\alpha_{i^*} > 0$ and the other two coefficients of ψ_{i^*} are bounded on $[y_1, y_2]$, there are $\kappa > 0$ and s_1 such that $\psi_{i^*}(s,y) \geq \kappa s^2$ for all $s \geq s_1$ and $y \in [y_1, y_2]$. Since $f_i(\bar{x} + yv) \leq 0$ and $t \geq \kappa s^2$, we obtain $f_i(x(s,t,y)) \leq sq_i^\mathsf{T}\bar{e} + \kappa s^2 q_i^\mathsf{T} d$, which is negative for all sufficiently large s.

Therefore, we have shown that there is some $\bar{s} \in \mathbb{R}$ such that

$$\{x(s,t,y) : s \geq \bar{s}, y \in [y_1, y_2], t \geq \psi_{i^*}(s,y)\} \subseteq S \cap H \subseteq \{x(s,t,y) : t \geq \psi_{i^*}(s,y)\}. \tag{9}$$

With all of this established, we now construct our objective function and mixed integer lattice. Let $\theta \in [y_1, y_2]$ be a rational number in the interior of the interval, and π_H denote the orthogonal projection onto H. Then for any $x \in \mathbb{R}^n$, we can define $(s(x), t(x), y(x))$ as the unique triple such that $x(s(x), t(x), y(x)) = \pi_H(x)$. We can then define our objective as

$$f_0(x) = t(x) - \psi_{i^*}(s(x), y(x)) + (y(x) - \theta)^2.$$

In other words, f_0 is defined through its restriction to H. The function f_0 is quadratic with rational coefficients. Indeed, $(s(x), t(x), y(x)) = (V_H^\mathsf{T} V_H)^{-1} V_H^\mathsf{T}(x - \bar{x})$, where V_H is the rational matrix with columns $\bar{e}$, d, and v. Moreover, ψ_{i^*} has rational coefficients, since $\bar{x}$, v, $\bar{e}$, d, and f_{i^*} are rational, and θ is rational. Now $f_0(x) \geq 0$ on $S \cap H$, since we showed $t \geq \psi_{i^*}(s,y)$ for each $x(s,t,y) \in S \cap H$. We construct a mixed integer lattice so that we can never attain 0, but are still able to approach it. Specifically, we note that the objective is 0 iff $y = \theta$ and $t = \psi_{i^*}(s,y)$, so we want to make this impossible.

Next, we write

$$\psi_{i^*}(s,y) = As^2 + B_1 sy + B_0 s + C(y),$$

where $A, B_1, B_0 \in \mathbb{Q}$, $A = \gamma_{i^*}\alpha_{i^*} > 0$, and C is a quadratic function with rational coefficients. Since $A > 0$, we can choose $\lambda \in \{1, 2\}$ such that $2A\lambda + B_1 \neq 0$. Let D be the least common denominator of the coefficients of $\psi_{i^*}(k + \lambda\theta, \theta)$ as a polynomial in k, which are rational since λ and θ are rational. The lattice is given by

$$\Lambda_H = \{x(k + \lambda\sigma, j + \frac{1}{2D}, \sigma) : k, j \in \mathbb{Z}, \sigma \in \mathbb{R}\}.$$

We extend Λ_H to a rational mixed integer lattice Λ in $\mathbb{R}^n$ by adding $n - 3$ linearly independent rational directions orthogonal to the linear subspace parallel to H, with $p - 2$ of them integer and $n - p - 1$ of them continuous. Then Λ has p integer components and $\Lambda \cap H = \Lambda_H$.

We must have $f_0(x) > 0$ on $S \cap H \cap \Lambda$, because in order for it to be equal to 0, we must have $\sigma = \theta$ and then $j + \frac{1}{2D} = \psi_{i^*}(k + \lambda\theta, \theta)$. But this is impossible, since $\psi_{i^*}(k + \lambda\theta, \theta)$ is an integer multiple of $\frac{1}{D}$ since $k \in \mathbb{Z}$, while $j + \frac{1}{2D}$ cannot be an integer multiple of $\frac{1}{D}$.

To show that we can approach 0, for $k \in \mathbb{Z}$ define $h_k(\sigma) = \psi_{i^*}(k + \lambda\sigma, \sigma)$, and compute the derivative

$$h_k'(\sigma) = (2A\lambda + B_1)k + 2A\lambda^2\sigma + 2B_1\lambda\sigma + \lambda B_0 + C'(\sigma).$$

Let $\nu := |2A\lambda + B_1| > 0$. For all k large enough, we have $|h_k'(\sigma)| \geq \frac{\nu k}{2}$ for all $\sigma \in [y_1, y_2]$, since the other terms are bounded independently of k.

Let $\epsilon_k = \frac{4}{\nu k}$, so for sufficiently large k we have $[\theta - \epsilon_k, \theta + \epsilon_k] \subseteq [y_1, y_2]$ and also $k + \lambda\sigma \geq \bar{s}$ for $\sigma \in [\theta - \epsilon_k, \theta + \epsilon_k]$. Now, the image $h_k([\theta - \epsilon_k, \theta + \epsilon_k])$ is an interval since h_k is continuous. By the mean value theorem and the lower bound on $|h_k'|$, this interval has length at least $2\epsilon_k \frac{\nu k}{2} = 4$, so it contains some $t_k \in \mathbb{Z} + \frac{1}{2D}$. Let $\sigma_k \in [\theta - \epsilon_k, \theta + \epsilon_k]$ be such that $h_k(\sigma_k) = t_k$, and define

$$z^k = x(k + \lambda\sigma_k, t_k, \sigma_k).$$

Then $z^k \in \Lambda$, and also $z^k \in S$ for sufficiently large k by (9). We can compute

$$\begin{aligned} f_0(z^k) &= t_k - \psi_{i^*}(k + \lambda\sigma_k, \sigma_k) + (\sigma_k - \theta)^2 \\ &= (\sigma_k - \theta)^2 \leq \epsilon_k^2, \end{aligned}$$

which tends to 0 as $k \to \infty$, showing the infimum is indeed 0, but not attained. □

Proof (of Corollary 4.11). By definition, every FW_n set is FW_0. Conversely, assume that S is FW_0. Let Λ be a rational mixed integer lattice, let P be a rational polyhedron, and let f_0 be a quadratic function with rational coefficients that is bounded from below on $S \cap P \cap \Lambda \neq \emptyset$. After the change of variables $x = \tau(y)$, we may assume that $\Lambda = \mathbb{Z}^p \times \mathbb{R}^{n-p}$. The transformed set is still FW_0, since τ is a rational affine bijection, and its affine hull A is still rational, of dimension $k \leq 2$.

We apply [9, Theorem 1.1] to the polyhedron A. Since $A \cap \Lambda$ is nonempty, it gives $p' \leq p, n'$, and a map $\sigma(z) = \tilde{x} + Gz$ with $\tilde{x} \in \mathbb{Q}^n$ and $G \in \mathbb{Q}^{n \times n'}$ of full column rank, such that $A = \sigma(A')$ for a full-dimensional polyhedron $A' \subseteq \mathbb{R}^{n'}$, and $A \cap \Lambda = \sigma(A' \cap \Lambda')$, where $\Lambda' := \mathbb{Z}^{p'} \times \mathbb{R}^{n'-p'}$. Since A is an affine subspace and σ is injective, we have $A' = \mathbb{R}^{n'}$ and $n' = k$.

Let $S' := \{z \in \mathbb{R}^k : f_i(\sigma(z)) \leq 0,\ i = 1, 2, \ldots, m\}$, so that $S = \sigma(S')$. Let $G^+ := (G^\mathsf{T} G)^{-1} G^\mathsf{T}$. The set S' is FW_0. Indeed, for a rational polyhedron $P'' \subseteq \mathbb{R}^k$ and a quadratic function g with rational coefficients, the map σ is a bijection from $S' \cap P''$ to $S \cap \sigma(P'')$, the set $\sigma(P'') = \{x \in A : G^+(x - \tilde{x}) \in P''\}$ is a rational polyhedron, and $g(z) = g(G^+(\sigma(z) - \tilde{x}))$. Since $k \leq 2$, Corollary 4.10 implies that S' is FW_k, where for $k = 0$ there is nothing to prove. Finally, $S \cap P \cap \Lambda = \sigma(S' \cap P' \cap \Lambda')$, where $P' := \sigma^{-1}(P)$ is a rational polyhedron, and $f_0 \circ \sigma$ is a quadratic function with rational coefficients. Hence f_0 attains its infimum on $S \cap P \cap \Lambda$. □

E Conditions on the objective

This appendix contains the proof of Theorem 5.1 and Corollary E.5.

The following lemma is the inductive step of our argument: it reduces the dimension by one. When we project out a recession direction d, the inequalities that do not depend on d are kept. The other inequalities can be dropped, because adding a large multiple of d to any point satisfies them.

Lemma E.1. *Consider an instance of* (1) *with rational data, where all constraints f_i $(i = 1, 2, \ldots, m)$ are convex and f_0 is bounded from above on $S \cap \Lambda$. Let $d \in \mathbb{Q}^n$ be a nonzero recession direction for all constraints, such that $f_0(x) = f_0(x + td)$ for every $x \in S$ and $t \in \mathbb{R}$.*

Then there exist $E \in \mathbb{Q}^{n \times (n-1)}$, $m' \in \mathbb{Q}^n$, $p' \in \{p - 1, p\}$, and a rational mixed integer lattice $\Lambda' \subseteq \mathbb{R}^{n-1}$ with p' integer components such that the problem

$$\begin{aligned} &\inf\ f_0(Ex + m') \\ &\text{s.t. } x \in S' \cap \Lambda', \end{aligned} \tag{10}$$

where $I := \{k \in \{1, 2, \ldots, m\} : q_k^\top d = 0\}$ *and*

$$S' := \{x \in \mathbb{R}^{n-1} : f_k(Ex + m') \leq 0, \quad k \in I\},$$

has the same infimum as the original instance of (1)*, and* $f_0(Ex + m')$ *is bounded from above on* $S' \cap \Lambda'$*. Moreover, the functions* $f_k(Ex + m')$*,* $k \in I$*, are convex. Finally, for every* $x \in S' \cap \Lambda'$ *there is* $\bar{x} \in S \cap \Lambda$ *with* $f_0(\bar{x}) = f_0(Ex + m')$*.*

Proof. Without loss of generality, we can assume $\Lambda = \mathbb{Z}^p \times \mathbb{R}^{n-p}$, because we can replace x with $\tau(y)$ and all assumptions are still met. Below, we also apply a rational linear change of variables that maps $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ onto itself. In the new variables, we construct the required map with $m' = 0$. Composing it with the two changes of variables gives the rational affine map $x \mapsto Ex + m'$ for the original instance.

Next, we bring d into a convenient form. Let $d = (d_\mathbb{Z}, d_\mathbb{R})$, scaled by a positive rational number so that $d_\mathbb{Z} \in \mathbb{Z}^p$ is primitive and $d_\mathbb{R} \in \mathbb{Q}^{n-p}$. If $d_\mathbb{Z} \neq 0$, let $U \in \mathbb{Z}^{p \times p}$ be a unimodular matrix such that $Ue^1 = d_\mathbb{Z}$. Then consider $T = \begin{pmatrix} U & 0 \\ D & I_{n-p} \end{pmatrix}$ where D has $d_\mathbb{R}$ as the first column and 0 for the rest of the entries, and I_{n-p} is the identity matrix. If $d_\mathbb{Z} = 0$, then $d_\mathbb{R}$ is nonzero, and instead let $U \in \mathbb{Q}^{(n-p)\times(n-p)}$ be an invertible matrix such that $Ue^1 = d_\mathbb{R}$. Then we can take $T = \begin{pmatrix} I_p & 0 \\ 0 & U \end{pmatrix}$, since no off-diagonal block is needed when $d_\mathbb{Z} = 0$.

In either case, T is a bijection from $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ to itself, so we can do a change of basis $x = Ty$ such that the lattice is still $\mathbb{Z}^p \times \mathbb{R}^{n-p}$, but our recession direction is now either e^1 or e^{p+1}, and all assumptions are still met.

Let $E \in \mathbb{Q}^{n\times(n-1)}$ be the matrix that inserts a zero component in position 1 or $p + 1$, respectively, and let $\Lambda' := \mathbb{Z}^{p'} \times \mathbb{R}^{n-1-p'}$, where $p' = p - 1$ if we remove an integer component and $p' = p$ if we remove a continuous component.

Now we prove that (1) and (10) have the same infimum. Let $x \in S' \cap \Lambda'$, which has objective value $f_0(Ex)$ for (10). Consider the corresponding point $Ex \in \mathbb{R}^n$.

By Lemma A.1, we have $Q_i d = 0$ and $q_i^\top d \leq 0$ for each $i = 1, 2, \ldots, m$. We already know that for every $k \in I$, $f_k(Ex) \leq 0$ since $x \in S' \cap \Lambda'$, while for every $k \in \{1, 2, \ldots, m\} \setminus I$, $Q_k d = 0, q_k^\top d < 0$. Thus,

$$\forall k \in \{1, 2, \ldots, m\} \setminus I, \exists t_0 > 0, \forall t \geq t_0, f_k(Ex + td) \leq 0.$$

Since d is the unit vector of the removed component, $Ex + td \in \mathbb{Z}^p \times \mathbb{R}^{n-p}$ for every $t \in \mathbb{Z}$. Moreover, $f_k(Ex + td) = f_k(Ex) \leq 0$ for every $k \in I$, since $Q_k d = 0$ and $q_k^\top d = 0$. Hence $Ex + td \in S \cap \Lambda$ for all sufficiently large $t \in \mathbb{Z}$. Since $Ex + td \in S$, the assumption implies that f_0 is constant on the line through $Ex + td$ with direction d. This line contains Ex, so $f_0(Ex + td) = f_0(Ex)$. Hence, for every such t, the point $\bar{x} := Ex + td$ lies in $S \cap \Lambda$ and satisfies $f_0(\bar{x}) = f_0(Ex)$. In particular, $f_0(Ex)$ is bounded from above on $S' \cap \Lambda'$, since f_0 is bounded from above on $S \cap \Lambda$.

For the other direction, take $x \in S \cap \Lambda$ and consider projecting out the chosen component. We can write the resulting point as x' for some $x' \in \mathbb{Z}^{p'} \times \mathbb{R}^{n-1-p'}$. Then $x' \in S' \cap \Lambda'$, because $Ex' = x + td$ for some $t \in \mathbb{R}$, and since d is a recession direction, Lemma A.1 implies $f_k(x) = f_k(x + td)$ for every $k \in I$, so $f_k(x) \leq 0$ iff $f_k(Ex') \leq 0$, and $f_0(Ex') = f_0(Ex' + td)$ by assumption, proving the infimum is the same.

Finally, convexity is preserved as $f_i(Ex)$ is the composition of a convex and linear function. □

The next two lemmas provide the remaining ingredients for Theorem 5.1.

Lemma E.2. *Consider an instance of* (1) *with rational data, where all constraints* f_i $(i = 1, 2, \ldots, m)$ *are convex. If* $S \cap \Lambda$ *is unbounded, then there is some* $d \in \mathbb{Q}^n \setminus \{0\}$ *which is a recession direction of all constraints.*

Proof. Let (x^k) be a sequence of points in $S \cap \Lambda$ with $\lim_{k\to\infty} \|x^k\| = \infty$. Replace (x^k) with a subsequence such that $\left(\frac{x^k}{\|x^k\|}\right)$ has a well defined limit, which we can do by compactness. Denote $d' = \lim_{k\to\infty} \frac{x^k}{\|x^k\|}$, and by applying Lemma A.3 to each constraint, d' must be a recession direction of each of them.

By Lemma A.1, the set of recession directions of all constraints is given by a polyhedron, which is rational since we assume all functions have rational coefficients. Since $\|d'\| = 1$, we can apply Lemma A.2 to find some $d \in \mathbb{Q}^n \setminus \{0\}$ which is a recession direction of all constraints. □

Lemma E.3. *Consider an instance of* (1) *with rational data, where all constraints f_i $(i = 1, 2, \ldots, m)$ are convex. Let $d \in \mathbb{Q}^n \setminus \{0\}$ be a recession direction of all constraints such that $f_0(x) = f_0(x + td)$ for every $x \in S \cap \Lambda$ and $t \in \mathbb{R}$.*

Then if we add $\pm\left(2d^\mathsf{T}Q_0x + q_0^\mathsf{T}d\right) \leq 0$ as constraints, $S \cap \Lambda$ remains unchanged, but now $f_0(x) = f_0(x + td)$ for every $x \in \tilde{S}$ and $t \in \mathbb{R}$, where $\tilde{S}$ is the continuous relaxation of the feasible set given by

$$\tilde{S} := \{x \in \mathbb{R}^n : 2d^\mathsf{T}Q_0x + q_0^\mathsf{T}d = 0, f_i(x) \leq 0, \quad i = 1, 2, \ldots, m\}.$$

Additionally, d is still a recession direction of these constraints, and they are both convex and rational.

Proof. By expansion we have

$$f_0(x + td) = t^2d^\mathsf{T}Q_0d + 2td^\mathsf{T}Q_0x + tq_0^\mathsf{T}d + x^\mathsf{T}Q_0x + q_0^\mathsf{T}x + c_0.$$

This is a constant function of t iff $d^\mathsf{T}Q_0d = 0$ and $2d^\mathsf{T}Q_0x + q_0^\mathsf{T}d = 0$.

By assumption $f_0(x + td)$ is constant as a function of t for each $x \in S \cap \Lambda$, so we must have $d^\mathsf{T}Q_0d = 0$ and for every $x \in S \cap \Lambda$, $2d^\mathsf{T}Q_0x + q_0^\mathsf{T}d = 0$, so adding the additional constraints does not change $S \cap \Lambda$. On the other hand, adding the constraint ensures that these two conditions are met for every $x \in \tilde{S}$, so $f_0(x + td)$ is constant as a function of t for each $x \in \tilde{S}$.

The direction d remains a recession direction of the new constraints, since $d^\mathsf{T}Q_0d = 0$. The new constraints are convex, since they are affine, and they have rational coefficients, since d, Q_0, and q_0 are rational. □

Proof (of Theorem 5.1). We proceed by induction on the dimension of the problem. If $n = 0$, then there is only a single feasible point since we assume the feasible region is nonempty, and it trivially attains the infimum of all feasible points.

For $n > 0$, if $S \cap \Lambda$ is bounded, since it is closed as the intersection of the closed sets S and Λ, it is compact, so by the extreme value theorem f_0 attains a minimum value on it.

Otherwise $S \cap \Lambda$ is unbounded, and we apply Lemma E.2 to get a rational recession direction d of all constraints. By assumption, f_0 is bounded both above and below on $S \cap \Lambda$. Therefore, for every $x \in S \cap \Lambda$, $f_0(x + td)$ must be a constant function of t, because $f_0(x + td)$ is a polynomial in t. Lemma A.4 implies there are arbitrarily large values of t such that $x + td \in S \cap \Lambda$, so if $f_0(x + td)$ were not a constant function of t, it would tend to either ∞ or $-\infty$ for large values of t, contradicting that it is bounded in both directions.

Then if we add the constraints $\pm\left(2d^\mathsf{T}Q_0x + q_0^\mathsf{T}d\right) \leq 0$ to our system, Lemma E.3 ensures we do not change $S \cap \Lambda$, but now we meet the conditions to apply Lemma E.1. This gives a lower dimensional problem with the same infimum, which by the induction hypothesis attains its infimum at some feasible point $x \in S' \cap \Lambda'$, and as noted in Lemma E.1, there is a corresponding point $\bar{x} \in S \cap \Lambda$ which attains the infimum of (1). □

Remark E.4. If $p = 0$, that is, $\Lambda = \mathbb{R}^n$, then Theorem 5.1 holds without the rationality assumptions. Indeed, rationality is used in Lemmas A.4, E.1, and E.2 to stay in the mixed integer lattice, and in Lemma E.3 to keep the added constraints rational for the next step of the induction. For $p = 0$, the limit direction in the

proof of Lemma E.2 can be used directly as the recession direction d, the change of variables in the proof of Lemma E.1 can be real, the points $x + td$ with $t \geq 0$ are in Λ, and the added constraints need not be rational.

For at most one integer variable, the continuous result used in Theorem 3.1 is Theorem 5.1 with $p = 0$. This gives the following corollary.

Corollary E.5. *Consider a problem of the form* (3)*, or its continuous version with $\Lambda = \mathbb{R}^n$, where all g_j $(j = 0, 1, \ldots, m)$ are quadratic functions, all constraint functions g_i $(i = 1, 2, \ldots, m)$ are convex, and g_0 is quasi-convex over $K := \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, 2, \ldots, m\}$. Then the problem attains its infimum (when it is finite).*

Proof. We first show that, for every set $K' \subseteq K$ defined by convex quadratic inequalities, the problem $\inf\{g_0(x) : x \in K'\}$ attains its infimum when it is finite. For $K' = K$, this is the continuous case. Let $\bar{y} + Wz$, $z \in \mathbb{R}^k$, be a parametrization of the affine hull of K', where $\bar{y} \in \mathbb{R}^n$ and $W \in \mathbb{R}^{n \times k}$ has full column rank. In the variables z, the feasible set is full-dimensional and defined by convex quadratic inequalities. The objective $\tilde{g}_0(z) := g_0(\bar{y} + Wz)$ is a quadratic function that is quasi-convex over the feasible set, since $K' \subseteq K$. Since all variables are continuous, this change of variables does not affect attainment. Let $\tilde{Q}$ be the symmetric matrix of the quadratic part of $\tilde{g}_0$. If $\tilde{Q}$ is positive semidefinite, we add the constraint $\tilde{g}_0(z) - r \leq 0$ for some r larger than the infimum, as in the proof of Corollary 5.2. Otherwise, [2, Theorem 6.3] shows that $\tilde{g}_0$ is bounded from above on the feasible set, as in the proof of Corollary 5.2. In both cases, the objective is bounded from above on the feasible region, so the infimum is attained by Theorem 5.1 with $p = 0$ and Remark E.4.

For the mixed integer case, we apply Theorem 3.1. As in the proof of Corollary 3.2, the set in its hypothesis is semialgebraic, so it has finitely many connected components. The feasible set of every relaxation (4) is $K \cap \{x \in \mathbb{R}^n : L \leq a^\mathsf{T} x \leq U\}$, which is defined by convex quadratic inequalities. Hence every relaxation with a finite infimum attains it. □