When does linking disjunctive blocks preserve exactness?

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Abstract

Strong formulations for assignment and network design often split each capacitated resource into one copy per level, the number of items it serves, and then link the levels of different resources. We ask when linking disjunctive blocks preserves exactness, which of their levels must be exact, and how the answer depends on the structure of the coupling. The linked formulation is ideal if and only if the links are integral when items, weights and capacities are ignored, and each block is exact at the levels the links actually allow. Sufficiency follows from disjunctive programming; necessity holds exactly for the levels used by an admissible profile, so links can exclude a bad level but never repair one they admit. Our main result characterizes when the formulation is sharp, that is, exact in the original variables. Integral links are then not needed: it suffices that they spread each resource's level distribution, in the convex order, around an admissible one, and this criterion is correct precisely when the level capacities satisfy a midpoint inequality, which tightened capacities can violate. Links are integral on forests and oriented sublattices; for a count relation an arc-flow network describes them, and a shortest path in it shows how cut-set inequalities designed for Steiner arborescences can cut off optimal solutions in unsplittable network design.

Keywords: Extended formulation, Disjunctive programming, Ideal and sharp formulations, Convex order, Dantzig–Wolfe decomposition, Cardinality-indexed formulation

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1. Introduction

Many strong extended formulations split each capacitated resource into disjunctive terms, one per value of a discrete attribute such as the number of items it serves [1, 2, 3, 4] or the value of a linear form [5], and then link the terms of different resources by what the rest of the model implies: an arc leaving a node of an arborescence carries no more commodities than enter it [3], the arcs of a cut carry at least the commodities that must cross it [4], a workload changes by at most Δ between periods. This paper answers one question about such formulations: *when does linking disjunctive blocks preserve exactness, which of their levels must be exact, and if exactness is lost, is the defect in the links or in the blocks?* We take exactness of the individual blocks at the relevant levels as given; how to recognize it for cardinality-indexed knapsack blocks is the subject of the companion manuscript [6].

The answer separates cleanly (Theorem 1). The relaxation equals its integer hull, that is, the formulation is *ideal* [7, 8], if and only if (i) the links, read as a polytope in the level variables alone, with all items, weights and capacities ignored, are integral; and (ii) each block is locally exact at every level that some admissible profile of the links uses. Sufficiency is a direct consequence of Balas's description of disjunctive hulls [9, 10], and we claim nothing new for it. Necessity is sharper than the standard sufficient conditions, which ask every term to be integral [9, 10, 5]: the links decide which levels matter, and, by a property of single cardinality-indexed blocks from [6], a failing occurring level remains visible in the original variables however the blocks are linked. So links that exclude a bad level restore exactness, whereas no links, however strong, repair a bad level they admit.

Main result: sharpness via convex order. In facility location and network design the objective and the remaining constraints see each resource only through its aggregated variables, so what matters is whether the formulation is *sharp* [7]: whether its relaxation projects onto the convex hull of the integer points in these variables. One might expect either that integral links remain necessary, as they are when every resource has at most two levels, or that nothing about the links is needed. Neither holds (Theorem 2). It suffices that the links only spread each resource's level distribution around an admissible one, in the convex order, and this criterion is correct exactly when the level capacities satisfy a midpoint inequality (M): for every capacity choice that violates (M), some coupling breaks it. Condition (M) holds for level-independent capacities but can fail for the tightened level capacities

used in [4] (Remark 1).

Checkable conditions and an application. Testing (i) directly requires the profile hull, which can be as hard to describe as the integer program. Integrality of pairwise links on forests and oriented sublattices follows from known results for graphical models; we add its consequence for exactness, covering workload bands and order constraints (Corollary 2, Proposition 1). For one count relation, an arc-flow network attached to the level variables describes the profile hull exactly and decides by a shortest path whether an inequality derived in one model remains valid in another (Corollaries 3 and 4). This gives a mechanism for a failure observed in [4], where cut-set inequalities designed for the Steiner arborescence formulation of [3] cut off an optimal solution of unsplittable network design (Section 5.3).

Theorems 1 and 2 are the main results: they say when linking preserves exactness in all variables (ideal) and in the original variables (sharp). Section 5 derives their consequences for pairwise links, count relations and the transfer of inequalities.

Relation to earlier work. Companion papers. The companion manuscript [6], under review at *Operations Research*, develops the polyhedral structure of a single cardinality-indexed knapsack block: it characterizes the fractional vertices of its levels, decides when the block is exact, and derives the consequences for the Dantzig–Wolfe bound when blocks interact only through aggregated variables. The present paper takes local exactness of the blocks as its starting point and asks when it is preserved once the level variables of different blocks are linked. We use from [6] only property (N) of Section 2, the characterization of local exactness by straddles, which serves to make examples concrete, and the load-level example of Section 3. When $\mathcal{P} = \prod_r \Delta_r$, Theorem 1(d) and Corollary 1 reduce to results of [6]; every result for a coupling that is a proper subset of $\prod_r \Delta_r$ is new here.

Disjunctive hulls and graphical models. Disjunctions coupled through an integral 0/1 polytope [9, 10] give sufficiency (Lemma 1); [5] studies disjunction over the value of a linear form, of which the item count is a case. Tree-structured marginal polytopes are integral while local relaxations on cycles are not [11], and the local relaxation is tight for pairwise functions supermodular on chains [12, Sec. 8] and for balanced binary models [13]. Since the indicator of a sublattice is supermodular, the integrality in Proposition 1 follows from [12]; what is new is its consequence for exactness, together with a short self-contained proof through the co-monotone coupling.

Convex order and state graphs. Convex order [14] and lower convex envelopes are classical, and a convex-combination model of a piecewise-linear cost relaxes to its lower convex envelope [15]. The count chain of Section 5.2 is a dynamic-programming state graph [16], an arc-flow network [17, 18] and an exact decision diagram [19, 20, 21]; we claim neither its construction nor its integrality, only what follows from attaching it to the level variables.

2. Setting

We follow [6]. Each block (resource) r in a finite set R' has eligible items K_r, positive integer weights d_{rk} and an integer capacity u_r, and $F_r^\circ \subseteq \{0,1\}^{K_r} \times \{0,1\}$ is the set of incidence vectors (χ, η) of feasible item sets served by an open resource, together with $(0,0)$. For a level $\ell \in \Lambda_r = \{1, \dots, L_r\}$ and an admissible level capacity $\bar{u}_{r\ell}$, the level polytope is $P_\ell^r = \{\chi \in [0,1]^{K_r} : |\chi| = \ell,\ d_r(\chi) \le \bar{u}_{r\ell}\}$. A level capacity is *admissible* if the integer points of P_ℓ^r are exactly the feasible ℓ-sets, that is, if it is at least the largest weight of a feasible ℓ-set and below the smallest weight of an infeasible one. Two natural choices are the capacity u_r itself, which does not depend on ℓ, and the *tightened* value $\bar{u}^{\mathrm{t}}_{r\ell}$, the largest weight of a feasible ℓ-set.

A level ℓ of block r is *locally exact* if P_ℓ^r is an integral polytope. This is the hypothesis on the individual blocks that our results take as input. By [6], a level is locally exact if and only if it has no *straddle*, a triple (A, i, j) with $|A| = \ell - 1$, $i, j \notin A$ and $d_{rj} < \bar{u}_{r\ell} - d_r(A) < d_{ri}$, whose vertex $\mathbf{1}_A + te_i + (1-t)e_j$, with $t \in (0,1)$ chosen so that the capacity is tight, is fractional; whether a level has one depends on the choice of $\bar{u}_{r\ell}$. We use straddles only to check local exactness in examples. Theorem 1(d) also uses the following property of cardinality-indexed blocks, established in [6, Thm. 2(b)]:

(N) if v is a fractional vertex of P_ℓ^r, then $(v, 1) \notin \mathrm{conv}(F_r^\circ)$,

that is, a fractional vertex of one level is not absorbed by the integer points of the block. Property (N) fails for load levels (Section 3).

We state the main result for general disjunctive blocks. Block r has a finite set Λ'_r of terms containing 0, and for each term a a nonempty polytope $T_r(a) \subseteq [0,1]^{K_r}$ with $T_r(0) = \{0\}$ that contains an integer point; a term is *locally exact* if $T_r(a)$ is integral. Its term variables are $x_{r\cdot a} \in \mathbb{R}^{K_r}$ and $y_{r,a} \ge 0$, $a \in \Lambda'_r$. Cardinality levels are the case $\Lambda'_r = \{0\} \cup \Lambda_r$ and $T_r(\ell) = P_\ell^r$; load levels are the case $T_r(q) = \{\chi \in [0,1]^{K_r} : d_r(\chi) = q\}$. Let $\Delta_r = \{y_r \in \mathbb{R}^{\Lambda'_r}_{\ge 0} :$

$\sum_a y_{r,a} = 1\}$, and for a profile $j = (j_r)_{r \in R'} \in \prod_r \Lambda'_r$ let $e_j = (e_{j_r})_r$ be its level indicator vector.

A *level coupling* is a polyhedron $\mathcal{P} \subseteq \prod_r \Delta_r$ in the level variables; it may be the projection of a system with auxiliary continuous variables. Its *profiles* are $\mathcal{J} = \{j : e_j \in \mathcal{P}\}$, and $Y = \operatorname{conv}\{e_j : j \in \mathcal{J}\}$ is its integer hull; throughout, $\mathcal{J} \neq \emptyset$. The relaxation and the integer model are

$$\mathcal{R} = \big\{(x, y) :\ x_{r\cdot a} \in y_{r,a} T_r(a)\ (r \in R',\ a \in \Lambda'_r),\ y \in \mathcal{P}\big\},$$
$$\mathcal{I} = \operatorname{conv}\big(\mathcal{R} \cap \{0,1\}^{\dim}\big).$$

For a term a of block r and $v \in T_r(a)$, let $\operatorname{emb}_r(a, v)$ be the vector of term variables of block r with $y_{r,a} = 1$, $x_{r\cdot a} = v$ and all other entries zero, and let

$$\mathcal{C} = \operatorname{conv}\Big(\bigcup_{j \in \mathcal{J}} \prod_{r \in R'} \operatorname{emb}_r\big(j_r, T_r(j_r)\big)\Big)$$

be the hull of the coupled terms. Then $\mathcal{I}$ is the same expression with $T_r(j_r)$ replaced by its integer points, so $\mathcal{I} \subseteq \mathcal{C}$. A term a of block r *occurs* if $a = j_r$ for some $j \in \mathcal{J}$. All statements are in the space of term variables, which is the space of the formulation and admits level-dependent costs. Table 1 collects the notation.

The main example is *pairwise coupling*. Let $G = (R', E)$ be a graph with admissible pairs $E_{rs} \subseteq \Lambda'_r \times \Lambda'_s$ for each edge, and variables $z_{rs}(a, b) \geq 0$ for $(a, b) \in E_{rs}$ with

$$\begin{aligned} \textstyle\sum_{b:(a,b)\in E_{rs}} z_{rs}(a,b) &= y_{r,a} \quad (a \in \Lambda'_r), \\ \textstyle\sum_{a:(a,b)\in E_{rs}} z_{rs}(a,b) &= y_{s,b} \quad (b \in \Lambda'_s). \end{aligned} \tag{1}$$

Let $\mathcal{P}_G$ be the projection onto y of $\prod_r \Delta_r$ intersected with (1). This is the local polytope of a graphical model, restricted to the support E_{rs}.

3. Ideal formulations: decomposition of exactness

Theorem 1 shows that $\mathcal{I} \subseteq \mathcal{C} \subseteq \mathcal{R}$, that the second inclusion is an equality exactly when the level coupling is integral, and that the first is an equality exactly when the occurring terms are integral. Exactness means $\mathcal{R} = \mathcal{I}$, equivalently that the relaxation and the integer model have the same optimal value for every linear objective in the term variables.

Table 1: Notation. Terms generalize levels; for cardinality levels $T_r(\ell) = P^r_\ell$.

$r \in R'$	block (resource)
K_r, d_{rk}, u_r	eligible items, weights, capacity
$\Lambda_r = \{1, \dots, L_r\}$, Λ'_r	levels; $\Lambda'_r = \{0\} \cup \Lambda_r$, 0 = closed
$\bar{u}_{r\ell}$, P^r_ℓ	level capacity, level polytope
$y_{r,\ell}$, $x_{r\cdot\ell}$	level selector, item vector at level ℓ
x_r, y_r	aggregated variables
Δ_r	level distributions of block r
$\mathcal{P} \subseteq \prod_r \Delta_r$	level coupling (the links)
$\mathcal{J}$, Y	admissible profiles, profile hull
$\mathcal{R} \supseteq \mathcal{C} \supseteq \mathcal{I}$	relaxation, coupled-term hull, integer hull

Lemma 1. $\{(x, y) : x_{r\cdot a} \in y_{r,a} T_r(a)\ \forall r, a,\ \ y \in Y\} = \mathcal{C}$.

Proof. The right-hand side is contained in the left, which is convex. Conversely, let $y = \sum_{j \in \mathcal{J}} p(j) e_j$ with p a probability distribution on $\mathcal{J}$, so that $\sum_{j : j_r = a} p(j) = y_{r,a}$ for all r, a. For $y_{r,a} > 0$ write $x_{r\cdot a} = y_{r,a} v_{ra}$ with $v_{ra} \in T_r(a)$; if $y_{r,a} = 0$ then $x_{r\cdot a} = 0$, and we pick any $v_{ra} \in T_r(a)$. Then the point

$$\textstyle\sum_{j \in \mathcal{J}} p(j) \left(\mathrm{emb}_r(j_r, v_{r j_r}) \right)_{r \in R'}$$

has entries $\sum_{j : j_r = a} p(j) = y_{r,a}$ and $\sum_{j : j_r = a} p(j) v_{ra} = x_{r\cdot a}$, so it equals (x, y). □

Theorem 1. *For disjunctive blocks with a level coupling $\mathcal{P}$:*

(a) $\mathcal{C} \subseteq \mathcal{R}$, $\mathrm{proj}_y \mathcal{R} = \mathcal{P}$ and $\mathrm{proj}_y \mathcal{I} = Y$.

(b) $\mathcal{R} = \mathcal{I}$ if and only if (i) $\mathcal{P} = Y$ and (ii) $\mathcal{C} = \mathcal{I}$.

(c) Condition (ii) holds if and only if every occurring term is locally exact.

(d) For cardinality levels, $\mathcal{R} = \mathcal{I}$ if and only if $\mathcal{P}$ is integral and every occurring level of every block is locally exact. If some occurring level is not locally exact, the projections of $\mathcal{R}$ and $\mathcal{I}$ onto the aggregated variables $x_r = \sum_a x_{r\cdot a}$, $y_r = \sum_{a \neq 0} y_{r,a}$ differ, so some linear objective in these variables has a strictly smaller relaxation value.

Proof. (a) $Y \subseteq \mathcal{P}$ because $e_j \in \mathcal{P}$ for $j \in \mathcal{J}$, so Lemma 1 gives $\mathcal{C} \subseteq \mathcal{R}$. Every $y \in \mathcal{P}$ extends to a point of $\mathcal{R}$ by choosing $x_{r\cdot a} = y_{r,a} v$ for any $v \in T_r(a)$,

and the integer points of $\mathcal{R}$ have $y = e_j$ with $j \in \mathcal{J}$. (b) If $\mathcal{P} = Y$, then $\mathcal{R} = \mathcal{C}$ by Lemma 1, which equals $\mathcal{I}$ by (ii). If $\mathcal{P} \neq Y$, the projections in (a) differ, so $\mathcal{R} \neq \mathcal{I}$. If $\mathcal{C} \neq \mathcal{I}$, then $\mathcal{I} \subsetneq \mathcal{C} \subseteq \mathcal{R}$. (c) If every occurring term is integral, each product $\prod_r \mathrm{emb}_r(j_r, T_r(j_r))$, $j \in \mathcal{J}$, is the convex hull of its integer points, so $\mathcal{C} = \mathcal{I}$. Conversely, suppose that an occurring term $T_r(a)$ is not integral, and let v be a fractional vertex of it. Because a occurs, there is a profile $j \in \mathcal{J}$ with $j_r = a$, and because every term contains an integer point, we may choose integer points $w_s \in T_s(j_s)$ for $s \neq r$. The point with block r equal to $\mathrm{emb}_r(a, v)$ and block s equal to $\mathrm{emb}_s(j_s, w_s)$ lies in $\mathcal{C}$ and has $y_{r,a} = 1$. The inequality $y_{r,a} \leq 1$ is valid for $\mathcal{I}$, so the points of $\mathcal{I}$ with $y_{r,a} = 1$ form a face of $\mathcal{I}$. As $\mathcal{I}$ is the convex hull of finitely many integer points, this face is the convex hull of those integer points with $y_{r,a} = 1$, and each of them has $x_{r\cdot a} \in T_r(a) \cap \mathbb{Z}^{K_r}$. Hence $x_{r\cdot a} \in \mathrm{conv}(T_r(a) \cap \mathbb{Z}^{K_r})$ on the face. A fractional vertex of $T_r(a)$ is not a convex combination of other points of $T_r(a)$, so v lies outside this set and $\mathcal{C} \neq \mathcal{I}$. Occurrence is needed: a term in no admissible profile is never selected, and its vertices are irrelevant. (d) The equivalence is (b) with (c). For the last statement, let $a = \ell$ be an occurring level with a fractional vertex v, and take the point constructed in (c): its level vector is e_j for a profile $j \in \mathcal{J}$ with $j_r = \ell$, so it satisfies $y \in \mathcal{P}$, and every $x_{s\cdot j_s}$ lies in $T_s(j_s)$, so it is a point of $\mathcal{R}$. It aggregates in block r to $(v, 1)$, which lies outside $\mathrm{conv}(F_r^\circ)$ by (N), whereas every point of $\mathcal{I}$ aggregates, block by block, into $\mathrm{conv}(F_r^\circ)$. □

Necessity of (i) is immediate from (a), since $\mathcal{R}$ and $\mathcal{I}$ project onto $\mathcal{P}$ and Y. The substance lies in (c) and (d), which go beyond the sufficient conditions of disjunctive programming in two ways. First, (c) is sharp in both directions: a fractional term that no admissible profile uses is never selected, so the coupling alone decides which terms must be integral, and a level that is not locally exact is harmless if no admissible profile uses it. Second, (d) shows that for cardinality levels the failure survives aggregation, so no objective in the original variables can hide it. This rests on (N) and fails for load levels, as the following example of [6] shows: take one block with weights $(1, 2)$, capacity 3 and load terms $q \in \{0, 1, 2, 3\}$. The terms $q = 1$ and $q = 2$ have the fractional vertices $(0, \frac{1}{2})$ and $(1, \frac{1}{2})$, so the relaxation is not exact in the term variables; but both points lie in the integer hull $\{(\chi, \eta) : 0 \leq \chi \leq \eta\mathbf{1},\ \eta \leq 1\}$ of the block, and the aggregated relaxation is exact.

The two conditions are independent. A single block with a level that is

not locally exact has $\mathcal{P} = \Delta_r$, which is integral, and is not exact by (d); Example 2 below has locally exact blocks and $\mathcal{P}$ not integral. Checking (i) directly requires the profile hull Y, which can be as hard to describe as the integer program itself; Section 5 therefore gives conditions that can be checked on the coupling.

Corollary 1. *Suppose the blocks of R' with coupling $\mathcal{P}$ satisfy (i) and (ii), and the remaining constraints of the model involve R' only through the aggregated variables. Then the LP bound equals the Dantzig–Wolfe bound of the decomposition in which R' together with $\mathcal{P}$ forms one subproblem. This bound is at least the Dantzig–Wolfe bound of the decomposition into single blocks with $\mathcal{P}$ in the master problem.*

Proof. The LP optimizes over the remaining constraints intersected with $\mathcal{R} = \mathcal{I}$, which is the Dantzig–Wolfe master over the convex hull of the subproblem. In the block decomposition, each block is convexified separately and $\mathcal{P}$ is imposed in the master; this set contains $\mathcal{I}$. □

4. Sharpness: aggregated objectives

In many models the objective and the remaining constraints see a block only through its aggregated variables $x_r = \sum_\ell x_{r\cdot\ell}$ and $y_r = \sum_{\ell\geq 1} y_{r,\ell}$: in single-source facility location, for instance, the cost $\sum_r (f_r y_r + \sum_k c_{rk} x_{rk})$ of opening facilities and assigning items, and the assignment constraints, involve only these variables, while the levels serve to strengthen the relaxation. Let π denote the projection onto $(x_r, y_r)_{r\in R'}$. We call $\mathcal{R}$ *aggregated-exact* if $\pi(\mathcal{R}) = \pi(\mathcal{I})$, that is, if the formulation is sharp for these variables [7]. Equivalently, for every polyhedron Γ in the aggregated variables and every linear objective in them, optimizing over $\mathcal{R} \cap \pi^{-1}(\Gamma)$ and over $\mathcal{I} \cap \pi^{-1}(\Gamma)$ gives the same value. Theorem 1(d) shows that (ii) is necessary for this. We now show that (i) is not, and identify what replaces it. Throughout, $L_r \geq 1$ is the largest level of block r.

For a block r let

$$\mathcal{K}_r = \big\{ h \in \mathbb{R}^{\Lambda'_r} : h_0 = 0,\ h_{\ell-1} - 2h_\ell + h_{\ell+1} \geq 0 \quad (2 \leq \ell \leq L_r - 1) \big\},$$

and $\mathcal{K} = \prod_r \mathcal{K}_r$: the level-cost vectors that vanish at level 0 and are convex on $\{1, \dots, L_r\}$; the value h_1 is free, so the step from level 0 to level 1 is

arbitrary. For $p, q \in \Delta_r$ we say that p is a *spread* of q if

$$\begin{aligned} p_0 = q_0, \qquad \textstyle\sum_\ell \ell\, p_\ell = \sum_\ell \ell\, q_\ell, \\ \textstyle\sum_\ell (\ell-k)_+\, p_\ell \geq \sum_\ell (\ell-k)_+\, q_\ell \quad (2 \leq k \leq L_r - 1). \end{aligned} \tag{2}$$

After normalizing the open levels, these are equal means and $\mathbb{E}(X-t)_+ \geq \mathbb{E}(X'-t)_+$ at the integers, hence for all real t by piecewise linearity: p is a mean-preserving spread of q in the convex order [14, Sec. 3.A]. For instance, on levels $\{1,2,3\}$, mass $\frac{1}{2}$ on each of levels 1 and 3 is a spread of level 2: the resource serves the same number of items on average, but alternates between light and heavy use.

Every $h \in \mathcal{K}_r$ satisfies

$$\begin{aligned} h_\ell = h_1 + (h_2 - h_1)(\ell - 1) \\ + \textstyle\sum_{k=2}^{L_r-1} (h_{k-1} - 2h_k + h_{k+1})\,(\ell - k)_+ \end{aligned} \tag{3}$$

for $1 \leq \ell \leq L_r$ (both sides agree at $\ell = 1, 2$ and have equal second differences) and $h_0 = 0$. So $\mathcal{K}_r$ is spanned linearly by $u = (0, 1, \ldots, 1)$ and $w = (0, 1, \ldots, L_r)$ plus the cone of the hinges $g_k = ((\ell - k)_+)_\ell$, $2 \leq k \leq L_r - 1$, and $\mathcal{K}_r^* = \{d : h \cdot d \geq 0 \ \forall h \in \mathcal{K}_r\} = \{d : u \cdot d = w \cdot d = 0,\ g_k \cdot d \geq 0\}$; d_0 is free. For $p, q \in \Delta_r$, $d = p - q$ has $\sum_\ell d_\ell = 0$, so $u \cdot d = 0$ gives $d_0 = 0$, and $p - q \in \mathcal{K}_r^*$ if and only if p is a spread of q. Consider the condition

(i′) every $y \in \mathcal{P}$ admits $y' \in Y$ such that y_r is a spread of y'_r for every $r \in R'$; equivalently, $\mathcal{P} \subseteq Y + \mathcal{K}^*$.

Like (i), condition (i′) depends only on which level profiles are admissible, and (i) implies (i′). If $L_r \leq 2$, then (2) forces $p = q$, because mass and mean determine a distribution on two levels; hence (i′) and (i) coincide when every block has at most two nonzero levels.

Lemma 2. *For a block r with cardinality levels and aggregated costs (c_r, f_r), let $\varphi_r(0) = 0$ and $\varphi_r(\ell) = f_r + \min\{c_r \cdot \chi : \chi \in P_\ell^r\}$ for $\ell \geq 1$.*

(a) Every $h \in \mathcal{K}_r$ equals φ_r for some (c_r, f_r).

(b) Let D_{rm} be the smaller of $\bar{u}_{rm}$ and the sum of the m largest weights in K_r. Then $\varphi_r \in \mathcal{K}_r$ for every (c_r, f_r) if and only if

$$(M) \qquad D_{r,\ell-1} + D_{r,\ell+1} \leq 2\bar{u}_{r\ell} \qquad (2 \leq \ell \leq L_r - 1).$$

Condition (M) holds if $\bar{u}_{r\ell}$ does not depend on ℓ or all weights in K_r are equal; for the tightened capacities it holds if and only if $\bar{u}^{\mathrm{t}}_{r\ell}$ is midpoint concave in ℓ.

Proof. (a) Put the increments $h_\ell - h_{\ell-1}$, which are nondecreasing, as costs on the items in order of nondecreasing weight; the ℓ lightest items fit whenever some ℓ items fit, so $\varphi_r(\ell) = h_\ell$ (details in the online appendix). (b) The weight $d_r(\chi)$ is continuous on the segment from a feasible m-set to the m heaviest items, which lies in $\{|\chi| = m,\ 0 \le \chi \le \mathbf{1}\}$, so D_{rm} is the largest weight in P^r_m. Fix $2 \le \ell \le L_r - 1$; the term f_r cancels in $\varphi_r(\ell-1) + \varphi_r(\ell+1) - 2\varphi_r(\ell)$. The set $Q = \frac{1}{2}P^r_{\ell-1} + \frac{1}{2}P^r_{\ell+1}$ lies in $\{|\chi| = \ell,\ 0 \le \chi \le \mathbf{1}\}$ with largest weight $\frac{1}{2}(D_{r,\ell-1} + D_{r,\ell+1})$, so (M) at ℓ says $Q \subseteq P^r_\ell$. If $Q \subseteq P^r_\ell$, then $\min_{P^r_\ell} c_r \cdot \chi \le \min_Q c_r \cdot \chi$, which is convexity at ℓ. If not, the costs $c_r = -d_r$ give $\varphi_r(\ell-1) + \varphi_r(\ell+1) - 2\varphi_r(\ell) = 2D_{r\ell} - D_{r,\ell-1} - D_{r,\ell+1} < 0$, since $D_{r\ell} \le \bar{u}_{r\ell}$. If $\bar{u}_{r\ell} = \bar{u}_r$, then $D_{rm} \le \bar{u}_r$; with equal weights w, $D_{rm} = mw$; for the tightened capacities, $D_{rm} = \bar{u}^{\mathrm{t}}_{rm}$. □

Let Φ_r be the set of vectors φ_r obtained from all aggregated costs (c_r, f_r), and $\Phi = \prod_r \Phi_r$; by Lemma 2, $\mathcal{K}_r \subseteq \Phi_r$ always, with equality exactly under (M). This is why (M) is the right boundary. Aggregated objectives see the levels only through the induced costs φ_r, and a spread raises the expected cost of every convex cost vector, by Jensen's inequality, so (i′) suffices exactly when every aggregated objective induces convex level costs. By the proof of Lemma 2(b), (M) says that $\frac{1}{2}P^r_{\ell-1} + \frac{1}{2}P^r_{\ell+1} \subseteq P^r_\ell$: alternating between light and heavy use never lets a resource carry more weight than steady use. When (M) fails, the costs $c_r = -d_r$ reward spreading, and Theorem 2(c) turns this into a counterexample.

Theorem 2. *Let the blocks have cardinality levels and let $\mathcal{P}$ be a level coupling.*

- *(a) If $\mathcal{R}$ is aggregated-exact, then (i′) and (ii) hold.*
- *(b) If (ii) holds, then $\mathcal{R}$ is aggregated-exact if and only if $\min_{y\in\mathcal{P}} \varphi \cdot y = \min_{y\in Y} \varphi \cdot y$ for every $\varphi \in \Phi$.*
- *(c) If every block satisfies (M), then $\mathcal{R}$ is aggregated-exact if and only if (i′) and (ii) hold. If a block r violates (M) at a level ℓ at which it is locally exact, the coupling of r alone given by $\mathcal{P} = \mathrm{conv}\{e_\ell, \frac{1}{2}e_{\ell-1} + \frac{1}{2}e_{\ell+1}\}$ satisfies (i′) and (ii) but is not aggregated-exact.*

(d) If $L_r \le 2$ for every block, then $\mathcal{R}$ is aggregated-exact if and only if $\mathcal{R} = \mathcal{I}$.

Proof. For $S \subseteq \prod_r \Delta_r$ let $A(S) = \pi\{(x,y) : y \in S,\ x_{r\cdot\ell} \in y_{r,\ell} P^r_\ell\ \forall r, \ell\}$. Then $\pi(\mathcal{R}) = A(\mathcal{P})$, and $\pi(\mathcal{I}) \subseteq \pi(\mathcal{C}) = A(Y)$ by Lemma 1, with equality under (ii) because then $\mathcal{C} = \mathcal{I}$ by Theorem 1(c). For aggregated costs (c, f) and fixed y, the minimum of $\sum_r (c_r \cdot x_r + f_r y_r)$ over the x in the definition of $A(S)$ separates into the terms $y_{r,\ell} \min\{c_r \cdot \chi : \chi \in P^r_\ell\}$. Hence

$$\min_{A(S)} \textstyle\sum_r (c_r \cdot x_r + f_r y_r) = \min_{y \in S} \varphi \cdot y, \qquad \varphi = (\varphi_r)_r. \tag{4}$$

(a) Condition (ii) is necessary by the last statement of Theorem 1(d). Suppose that (i′) fails, and let $p \in \mathcal{P} \setminus (Y + \mathcal{K}^*)$. The set $Y + \mathcal{K}^*$ is a polytope plus a polyhedral cone, hence a closed convex polyhedron, and strict separation gives h with $h \cdot p < \inf\{h \cdot q : q \in Y + \mathcal{K}^*\}$; in particular the infimum is finite. Since $Y + \mathcal{K}^*$ contains $y + td$ for all $y \in Y$, $d \in \mathcal{K}^*$, $t \ge 0$, finiteness forces $h \cdot d \ge 0$ on $\mathcal{K}^*$, so the infimum is $\min_{y \in Y} h \cdot y$, attained with $d = 0$. Block by block, $\mathcal{K}^*_r = \{d : u \cdot d = w \cdot d = 0,\ g_k \cdot d \ge 0\}$, so by Farkas' lemma $h_r = \alpha u + \beta w + \sum_k \lambda_k g_k$ with $\lambda \ge 0$; this vector vanishes at level 0 and has second differences $\lambda_k \ge 0$, so $h \in \mathcal{K}$. By Lemma 2(a), applied block by block, $h = \varphi$ for some aggregated costs; this step needs no hypothesis on the level capacities, so (a) holds for every admissible choice. By (4), the relaxation value is at most $h \cdot p$, whereas the value over $\pi(\mathcal{I}) \subseteq A(Y)$ is at least $\min_Y h \cdot y$. So $\mathcal{R}$ is not aggregated-exact. (b) Under (ii) we have $\pi(\mathcal{R}) = A(\mathcal{P})$ and $\pi(\mathcal{I}) = A(Y)$. Two polytopes coincide if and only if their support functions agree in every direction, and by (4) the support functions in direction (c, f) are $\min_{\mathcal{P}} \varphi \cdot y$ and $\min_Y \varphi \cdot y$, where $\varphi \in \Phi$ is induced by (c, f). (c) Necessity is (a). Conversely, (ii) holds, and by (b) it suffices to show $\min_{\mathcal{P}} \varphi \cdot y = \min_Y \varphi \cdot y$ for every $\varphi \in \Phi$. By Lemma 2(b), $\varphi \in \mathcal{K}$. For $y \in \mathcal{P}$ write $y = y' + d$ with $y' \in Y$ and $d \in \mathcal{K}^*$; then $\varphi \cdot y \ge \varphi \cdot y'$, and the reverse inequality holds because $Y \subseteq \mathcal{P}$. For the second statement, $Y = \{e_\ell\}$, every point of $\mathcal{P}$ is a spread of e_ℓ, and only level ℓ occurs, so (i′) and (ii) hold. With $c_r = -d_r$ and $f_r = 0$, the relaxation value is at most $-\frac{1}{2}(D_{r,\ell-1} + D_{r,\ell+1}) < -\bar{u}_{r\ell} \le -D_{r\ell} = \min_Y \varphi \cdot y$, which by (b) is the value over $\pi(\mathcal{I})$. (d) If $L_r \le 2$ for all r, then (i′) is (i), so by (a) aggregated exactness implies (i) and (ii), hence $\mathcal{R} = \mathcal{I}$ by Theorem 1(d). The converse is trivial. Lemma 2(b) is not used, so (d) holds for any admissible level capacities. □

This does not contradict Theorem 1: integral links remain necessary for the formulation to be ideal, that is, exact in the term variables, and Theorem 2 shows that for sharpness in the aggregated variables they can be replaced by the weaker condition (i′). Condition (i) is thus needed for aggregated objectives when every block has at most two nonzero levels, as in Corollary 2 and Example 2 below, but not in general, because convex induced costs penalize spreading. Part (b) holds for any admissible capacities. The witness coupling in (c) is minimal on purpose, but the phenomenon is not artificial: level capacities that violate (M) arise as the tightened capacities $\bar{u}^{\mathrm{t}}_{r\ell}$, which [4] uses on every arc. Remark 1 gives an instance; for such formulations the convex-order test of (c) must be replaced by the support-function test of (b).

Example 1. On a triangle, give each block three items of weight 1 and capacity 3, so $L_r = 3$ and every level is locally exact. Admit only the pairs $(1,3)$, $(3,1)$ and $(2,2)$ on every edge. Equations (1) give $y_{r,1} = y_{s,3}$, $y_{r,3} = y_{s,1}$ and $y_{r,2} = y_{s,2}$ on every edge, so around the odd cycle $y_{r,1} = y_{r,3} = t$ and $y_{r,2} = 1 - 2t$ for all r, with $t \in [0, 1/2]$. Hence $\mathcal{J} = \{(2,2,2)\}$ and $\mathcal{P}_G \neq Y = \{e_{(2,2,2)}\}$. With a level cost M on level 2 of every block, an objective in the term variables, the relaxation value is 0 at $t = 1/2$ and the integer optimum is $3M$, so $\mathcal{R} \neq \mathcal{I}$. Every point of $\mathcal{P}_G$ is, block by block, a spread of level 2, so $\mathcal{R}$ is aggregated-exact by Theorem 2(c).

Remark 1 (Tightened level capacities)**.** Tightened capacities need not be midpoint concave. In Example 1, replace each block by items of weights $(3,1,1,1)$ and costs $(-15,-10,-10,-10)$ with capacity 3 and the tightened level capacities $(\bar{u}_{r1}, \bar{u}_{r2}, \bar{u}_{r3}) = (3,2,3)$, which violate (M): $D_{r1} + D_{r3} = 6 > 4$. Every level is locally exact, but the induced costs $\varphi_r = (0,-15,-20,-30)$ are not convex on $\{1,2,3\}$, and the relaxation value -67.5 lies below the integer optimum -60 although (i′) holds; Theorem 2(b) detects the gap through this φ. With the common capacity 3, level 2 has a straddle and is not locally exact, in line with Theorem 2(c).

5. Consequences: checkable couplings and inequality transfer

By Theorems 1 and 2, once the blocks are known to be locally exact, exactness is a question about the level coupling alone: is $\mathcal{P}$ integral, or does it satisfy (i′)? Every result in this section has this form: a certificate on the coupling, namely a forest, oriented sublattices or an arc-flow chain, that

gives $\mathcal{P} = Y$ without reference to items, weights or capacities, combined with Theorem 1 for ideality or Theorem 2 for sharpness. Example 1 shows the other route, an odd cycle with $\mathcal{P} \neq Y$ that is sharp because (i′) holds. Corollary 4 uses a certificate in the opposite direction: since the chain describes Y_{cnt} exactly, validity of an inequality is a shortest-path question.

5.1. Pairwise coupling

Corollary 2. *If G is a forest, $\mathcal{P}_G$ is integral, so pairwise level coupling of cardinality-indexed blocks along a forest is exact if and only if every occurring level is locally exact. If G has a cycle, there are locally exact blocks and admissible pairs with $\mathcal{P}_G$ not integral and the relaxation value strictly below the integer optimum, for an objective in the aggregated variables.*

Proof. For a forest, a point (y, z) satisfying (1) is the pair of marginals of the distribution $p(j) = \prod_r y_{r,j_r} \prod_{\{r,s\}\in E} z_{rs}(j_r, j_s)/(y_{r,j_r} y_{s,j_s})$ on $\mathcal{J}$, with factors set to 0 when a denominator vanishes; this is the classical junction-tree argument [11], verified by summing out leaves. Hence $y \in Y$. For the second statement, Section A.1 of the online appendix constructs, on any cycle, blocks with two unit-weight items and pair sets that force adjacent open blocks to alternate between levels 1 and 2 and admit no all-open profile; the point with $y_{r,1} = y_{r,2} = \frac{1}{2}$ on the cycle then beats the integer optimum by at least 1, with item costs only. □

The cyclic witness is deliberately minimal: a single resource followed over time is a path, hence a forest, and bands, order constraints and monotone ties are covered on any graph by Proposition 1. The gap of the witness is visible to aggregated objectives; by Theorem 2(d), this is always so when every block has at most two nonzero levels, but not in general (Example 1).

Example 2. On a triangle with the blocks of the proof, admit the pairs $\{(a, b) : a \neq b\} \subseteq \{0, 1, 2\}^2$ on every edge. The profiles are the permutations of $(0, 1, 2)$, so every $y \in Y$ has $\sum_r y_{r,0} = 1$, whereas $y_{r,1} = y_{r,2} = 1/2$ lies in $\mathcal{P}_G$. The relaxation value is $-9/2$ and the integer optimum is -3.

On a cycle, integrality depends on how the admissible pairs fit together. Order the levels of each block r either increasingly or decreasingly; call this its *orientation*. A pair set E_{rs} is a *sublattice* for given orientations of r and s if, with $\wedge$ and $\vee$ the componentwise minimum and maximum in the oriented orders, $(a, b), (a', b') \in E_{rs}$ implies $(a, b) \wedge (a', b') \in E_{rs}$ and $(a, b) \vee (a', b') \in E_{rs}$.

Proposition 1. *If the blocks can be oriented so that every E_{rs} is a sublattice, then $\mathcal{P}_G$ is integral, for any graph G. Pairwise level coupling of such blocks is then exact if and only if every occurring level is locally exact.*

Proof. Reverse the level labels of every block with decreasing orientation, so that all orders are increasing; sublattices are preserved. Let (y, z) satisfy (1), and write $F_r(a) = \sum_{a' \le a} y_{r,a'}$. Fix an edge $\{r, s\}$. Among all nonnegative z' supported on E_{rs} with the marginals y_r and y_s, a nonempty polytope, choose one minimizing $\sum_{a,b} z'(a,b)(a-b)^2$. If its support contained (a, b') and (a', b) with $a < a'$ and $b < b'$, moving the smaller of the two masses, ε, to their meet (a, b) and join (a', b'), which lie in E_{rs}, would keep the marginals and change the objective by $-2\varepsilon(a'-a)(b'-b) < 0$. So any two points of the support are comparable, and a distribution supported on a chain is determined by its marginals: it is the co-monotone coupling, with $\Pr(a \le \alpha, b \le \beta) = \min\{F_r(\alpha), F_s(\beta)\}$. Hence the co-monotone coupling of y_r and y_s is supported on E_{rs}. Now let U be uniform on $[0, 1)$ and $j_r(U) = \min\{a : F_r(a) > U\}$ for every block. For each edge, $(j_r(U), j_s(U))$ has the co-monotone law, so it lies in E_{rs} for almost every U; thus $j(U) \in \mathcal{J}$ almost surely. The map $U \mapsto j(U)$ is piecewise constant with finitely many pieces, and y_r is the law of $j_r(U)$, so y is a convex combination of the vectors $e_{j(U)}$ and lies in Y. The second statement follows from Theorem 1(d). □

Whether suitable orientations exist reduces to balance of a signed graph [22] and is decided in time linear in $|R'| + |E|$ once each edge is classified, in $O(|E_{rs}|^2)$ time (online appendix, Section A.4).

The condition covers bands $\{(\ell, \ell') : |\ell - \ell'| \le \Delta\}$, order constraints $\{(\ell, \ell') : \ell \le \ell' + c\}$ and monotone ties $\{(\ell, g(\ell))\}$ with g nondecreasing, all positive. In multi-period assignment with block (i, t) for machine i in period t, level 0 meaning idle, workload smoothing over bands therefore attains, by Corollary 1, the Dantzig–Wolfe bound of the decomposition in which all coupled blocks form one subproblem whenever every block is locally exact at the occurring levels, also when bands link machines of the same period. The rule of [3] that an arc leaving a Steiner node carries at most as many commodities as enter it is an order constraint; that formulation also links levels through flow conservation, which involves many arcs and the item variables and is not of the form (1). The condition is sufficient, not necessary; Example 1 shows the failure it excludes: each edge admits the decreasing chain $\{(1,3),(2,2),(3,1)\}$, a sublattice only for opposite orientations, and the three negative edges of the triangle form an unbalanced cycle.

5.2. Count relations

Let (D) be a cardinality-indexed formulation whose blocks are coupled by a polyhedron Γ on aggregated variables, as in [6], and suppose that Γ implies $\sum_{r\in R'} \sigma_r \sum_{k\in K_r} x_{rk} = n_0$ for integers σ_r, n_0. Assignment constraints give $\sum_r \sum_k x_{rk} = |K|$; flow conservation summed over commodities gives a node count equation with $\sigma_r = \pm 1$ for entering and leaving arcs. Since $\sum_k x_{rk} = \sum_\ell \ell\, y_{r,\ell}$ on every block,

$$\sum_{r\in R'} \sigma_r \sum_\ell \ell\, y_{r,\ell} = n_0 \tag{5}$$

holds on the relaxation of (D), and every inequality in y that is valid for the binary solutions of (5) with $y_r \in \Delta_r$ is valid for (D). Multi-arc inequalities in the level variables, such as the cardinality matching and count-based cut-set inequalities of [1, 2, 3, 4], are of this kind when their validity follows from one count relation and $y_r \in \Delta_r$.

Order $R' = (r_1, \dots, r_m)$, let $\mathcal{S}_0 = \{0\}$ and $\mathcal{S}_k = \{s + \sigma_{r_k}\ell : s \in \mathcal{S}_{k-1},\ \ell \in \Lambda'_{r_k}\}$. The *count chain* is the unit flow from state $(0,0)$ to state (m, n_0) in the acyclic graph with an arc from $(k-1, s)$ to $(k, s+\sigma_{r_k}\ell)$ for each $s \in \mathcal{S}_{k-1}$ and $\ell \in \Lambda'_{r_k}$, with flow $\phi_k(s,\ell) \geq 0$, linked to the levels by

$$\sum_{s\in\mathcal{S}_{k-1}} \phi_k(s,\ell) = y_{r_k,\ell} \qquad (k = 1, \dots, m,\ \ell \in \Lambda'_{r_k}). \tag{6}$$

For a covering relation $\sum_r \sum_k x_{rk} \geq n_0$, replace $s+\ell$ by $\min\{s+\ell, n_0\}$. With $\bar\sigma = \max_r |\sigma_r|$ and $L_r \leq L$, the chain has $O(\bar\sigma m^2 L^2)$ arcs, and $O(m(n_0+1)L)$ when all $\sigma_r > 0$ (online appendix, Section A.3).

Corollary 3. *Let $\mathcal{J}_{\mathrm{cnt}}$ be the set of level profiles $(j_r)_{r\in R'}$, $j_r \in \Lambda'_r$, with $\sum_r \sigma_r j_r = n_0$, and $Y_{\mathrm{cnt}} = \mathrm{conv}\{e_j : j \in \mathcal{J}_{\mathrm{cnt}}\}$ their level-profile hull.*

(a) Extended formulation. *The projection onto y of the chain with* (6) *is Y_{cnt}. In particular, every inequality in y that is valid for the binary solutions of* (5) *with $y_r \in \Delta_r$ is implied by the relaxation of (D) together with the chain, for arbitrary blocks, since the chain involves only the level variables.*

(b) Exactness. *If the blocks of R' are locally exact at the levels occurring in $\mathcal{J}_{\mathrm{cnt}}$, the blocks with the chain describe the convex hull of their integer points whose levels satisfy* (5)*; otherwise they do not.*

(c) Decomposition bound. *Under the hypothesis of (b), the relaxation of (D) with the chain attains the Dantzig–Wolfe bound of the decomposition in which R' together with* (5) *forms one subproblem. This bound is at least the block-wise Dantzig–Wolfe bound, and the difference can be arbitrarily large (Example 3) or zero.*

The same holds for covering relations, with $\sum_r j_r \geq n_0$ in place of (5).

Proof. The unit-flow polytope of an acyclic graph is integral with the source–sink paths as vertices, and a path chooses exactly one level per block with $\sum_k \sigma_{r_k} \ell_k = n_0$; conversely every such profile is a path. The linear map (6) sends a path to the indicator vector of its profile, so the projection is Y_{cnt}; the inequalities in (a) are valid for every e_j, $j \in \mathcal{J}_{\mathrm{cnt}}$, hence for Y_{cnt}. Since Y_{cnt} is integral, (b) is Theorem 1(d) with $\mathcal{P} = Y_{\mathrm{cnt}}$, and (c) follows from Corollary 1. In the covering chain, states are truncated at n_0, and a path ends in state n_0 exactly when its levels sum to at least n_0. □

Unlike an arc-flow model or decision diagram, the chain tracks a count *across* resources and is attached to the level variables by (6) rather than replacing them; several chains sharing blocks form a consistent path problem [23], whose relaxation is in general weaker than the hull. Corollary 3(c) attains a Dantzig–Wolfe bound with a formulation written down in advance, complementing its recovery by cutting planes [24].

Example 3. Two facilities each serve up to three of four unit-weight items, every item is assigned to exactly one facility, and levels carry costs $f_{1,2} = f_{2,1} = f_{2,3} = M$ and $f_{r,\ell} = 0$ otherwise. The blocks are locally exact, so by Corollary 1 with $\mathcal{P} = \prod_r \Delta_r$ the relaxation of (D) equals the block-wise Dantzig–Wolfe bound; it is 0, attained by $y_{1,1} = y_{1,3} = 1/2$ and $y_{2,2} = 1$, which satisfies (5) with $n_0 = 4$. The admissible profiles are $(1, 3)$, $(2, 2)$ and $(3, 1)$, each of cost M, and the chain forces $y_{2,\ell} = y_{1,4-\ell}$, so its bound is M, the integer optimum. Thus the chain bound can exceed the block-wise Dantzig–Wolfe bound by an arbitrary amount; Figure 1 shows the chain.

The chain also makes transfers between models precise.

Corollary 4. *Let $\alpha \cdot y \geq \beta$ be an inequality in the level variables of R'.*

(a) *It is valid for the binary solutions of* (5) *with $y_r \in \Delta_r$ if and only if the shortest $(0, 0)$–(m, n_0) path in the count chain, with length $\alpha_{r_k,\ell}$ on the*

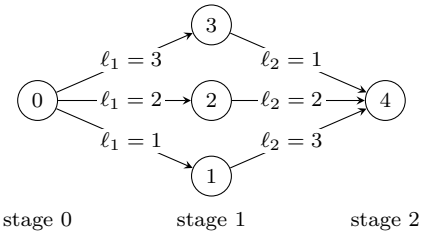


Figure 1: The count chain of Example 3 after deleting states on no source–sink path. An arc into stage k is labelled with the level of facility k; the three paths are the admissible profiles $(1, 3)$, $(2, 2)$ and $(3, 1)$, and the flow on them is the joint distribution of the two levels.

arcs of stage k and level ℓ, has length at least β. The same holds for a covering relation with the covering chain. Each test takes time linear in the number of arcs.

(b) Suppose the inequality is valid for (5)*, and a second model implies the covering relation $\sum_r \sum_\ell \ell\, y_{r,\ell} \geq n_0$ but not* (5)*. If the shortest path in the covering chain has length at least β, the inequality is valid for the second model. Otherwise that path is a level profile that satisfies the covering relation and violates the inequality, so validity in the second model requires an argument that excludes this profile.*

Proof. (a) By Corollary 3(a), validity for the binary solutions is validity on Y_{cnt}, respectively on its covering analogue. The minimum of $\alpha \cdot y$ over this set is attained at a vertex, which is the profile vector e_j of a source–sink path, and $\alpha \cdot e_j$ is the length of that path; the chain is acyclic. (b) Apply (a) to the covering chain; a shortest path is a profile that satisfies the covering relation. □

5.3. Application: cut-set inequalities in unsplittable network design

In the Steiner arborescence formulation of [3], each node other than the root is entered by exactly one arc, and the levels of the arcs of a minimal cut add up exactly to the number of terminals it separates. Multi-arc inequalities such as the p-arc cardinality matching inequalities of [3] are derived from this exact relation. In unsplittable multicommodity network design [4], the level of an arc is the number of commodities it carries. For a node set S, every commodity in $K(S) = \{k : o(k) \in S,\ t(k) \notin S\}$ crosses $\delta^+(S)$, but it may

cross more than once, and commodities that only pass through S also occupy its outgoing arcs. Only the covering relation

$$\sum_{a\in\delta^+(S)} \sum_{\ell} \ell\, y_{a,\ell} \geq |K(S)| \tag{7}$$

therefore holds. The paper [4] reports that a direct transfer of the inequalities of [3] cut off the optimal solution of one of its instances. Corollary 4(b) gives a mechanism for such failures and a test: one shortest path in the covering chain of (7) over $\delta^+(S)$. If it is shorter than β, it is an explicit profile of the cut that the inequality wrongly excludes. The following small network, which is not the instance of [4], illustrates the mechanism.

Example 4. Take nodes v, s, t, an arc (v, s) and two parallel arcs a, b from s to t, all of capacity 2, two commodities from s to t and two from v to t, all of demand 1, and $S = \{s\}$. Then $\delta^+(S) = \{a, b\}$, $|K(S)| = 2$ and $\Lambda'_a = \Lambda'_b = \{0, 1, 2\}$. Under the exact relation $\ell_a + \ell_b = 2$, the profiles are $(0, 2)$, $(1, 1)$ and $(2, 0)$, so $y_{a,2} + y_{b,2} \leq 1$ is valid. In the covering chain of (7), with length -1 on the arcs of level 2, the shortest path has length $-2 < -1$ and is the profile $(2, 2)$. It is realized: all four commodities cross from s to t on a or b, so every feasible solution puts both arcs at level 2, and the transferred inequality makes the instance infeasible.

Conversely, by Corollary 3(a) one chain per cut implies all inequalities valid under the exact relation at once.

6. Conclusion

The paper's message is the contrast between two kinds of exactness. A linked formulation of disjunctive blocks is ideal exactly when the links are integral and every level they allow is locally exact; links can exclude a bad level but never repair one they admit. It is sharp, exact in the original variables, under the weaker requirement that the links only spread level distributions in the convex order, and a midpoint inequality on the level capacities is exactly what makes this weakening valid. For count relations, one arc-flow chain per relation describes the links exactly and turns the transfer of inequalities between models into a shortest-path test. Four questions remain open: an item-free characterization of aggregated exactness when the capacities violate the midpoint inequality; a characterization of integral pairwise couplings beyond forests and sublattices; conditions under which several count chains

sharing blocks yield an integral coupling, which would cover the node count relations that flow conservation implies in the Steiner arborescence formulation of [3]; and an analogue of Theorem 2 for load levels.

Declarations

CRediT authorship contribution statement. **Md Shahrukh Anjum:** Conceptualization, Methodology, Formal analysis, Software, Validation, Writing – original draft, Writing – review & editing.

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Data availability. Scripts that verify Corollaries 2 and 3, Proposition 1, Lemma 2, Theorem 2 and Examples 2–4, exactly or on random instances, are provided as supplementary material; the online appendix lists them.

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process. During the preparation of this work the author used Claude (Anthropic) to assist with drafting and revising the text, checking proofs, and writing verification code. After using this tool, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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Online appendix to "When does linking disjunctive blocks preserve exactness?"

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Numbers without the prefix A refer to the main text.

A.1. Proof of the second statement of Corollary 2

Let G contain a cycle $r_1, \dots, r_k$, $k \geq 3$. Give every block on the cycle two items of weight 1, capacity 2 and item cost -1, so it is straddle-free with levels $\{1, 2\}$, and give blocks off the cycle zero costs. The admissible pairs are as follows (Figure A.1):

- on $\{r_i, r_{i+1}\}$ for $i = 1, \dots, k-1$: $(1, 2)$ and $(2, 1)$, and on $\{r_1, r_2\}$ also $(0, 1)$ and $(0, 2)$;
- on the closing edge $\{r_k, r_1\}$: $(1, 2)$ and $(2, 1)$ if k is odd, $(1, 1)$ and $(2, 2)$ if k is even, and in both cases also $(1, 0)$ and $(2, 0)$;
- on every other edge of G: all pairs.

Only r_1 may be closed. If all blocks of the cycle are open, the levels alternate along $r_1, \dots, r_k$, so r_k has the same level as r_1 if k is odd and the other level if k is even, which the closing edge forbids in both cases. Hence every profile closes r_1, and closing r_1 and alternating $1, 2$ along $r_2, \dots, r_k$ gives a profile, so $\mathcal{J} \neq \emptyset$. In a profile, $r_2, \dots, r_k$ are open and alternate, so at most $\lceil (k-1)/2 \rceil$ of them take level 2, and this is attained by starting with level 2 at r_2. Since a block at level ℓ costs at least $-\ell$, with equality for its unique feasible ℓ-set, the integer optimum on the cycle is $-(k-1) - \lceil (k-1)/2 \rceil$, which equals $-3k/2 + 3/2$ for odd k and $-3k/2 + 1$ for even k.

Now consider the point with $y_{r,0} = 0$, $y_{r,1} = y_{r,2} = 1/2$, $x_{r\cdot 1} = (1/4, 1/4) \in (1/2)P_1^r$ and $x_{r\cdot 2} = (1/2, 1/2) \in (1/2)P_2^r$ on the cycle. On each cycle edge, put $z = 1/2$ on the two admissible pairs with both levels positive and $z = 0$ on the pairs involving level 0. Every level of either endpoint then appears in exactly one positive pair, so each equation (1) reads $1/2 = 1/2$ or $0 = 0$. On

the remaining edges put $z_{rs}(a,b) = y_{r,a}y_{s,b}$, and off the cycle choose any point of the relaxation. The cost is $-3k/2$, at least 1 below the integer optimum. The objective uses only item costs, so the gap is also visible to objectives in the aggregated variables. The pair sets on the two edges at r_1 are sublattices for neither orientation, so Proposition 1 does not apply.

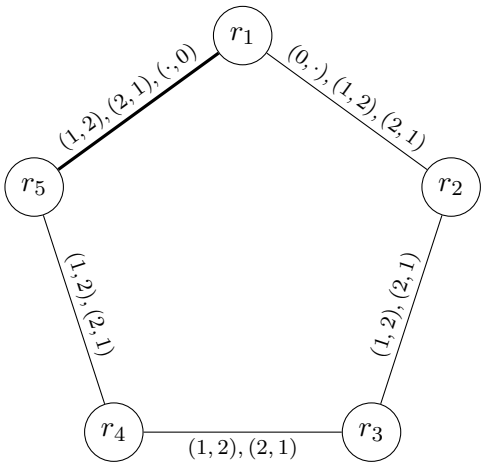


Figure A.1: The construction for $k = 5$. Interior edges force adjacent open blocks to alternate between levels 1 and 2; around an odd cycle the closing edge (thick) then admits no all-open profile, so every profile closes r_1. The point with $y_{r,1} = y_{r,2} = 1/2$ on every block satisfies all edge equations. Pairs are written as (level of the first-named block, level of the second); $(0,\cdot)$ stands for $(0,1),(0,2)$ and $(\cdot,0)$ for $(1,0),(2,0)$.

A.2. Proof of Lemma 2(a)

(a) Every level $1,\dots,L_r$ has a feasible set, because P^r_ℓ contains an integer point, and $|K_r| \geq L_r$. Let $\delta_\ell = h_\ell - h_{\ell-1}$ for $2 \leq \ell \leq L_r$, which is nondecreasing in ℓ. Order the items by nondecreasing weight as $k_1, k_2, \dots$, and set $c_{rk_1} = \delta_2$, $c_{rk_\ell} = \delta_\ell$ for $2 \leq \ell \leq L_r$, $c_{rk} = \delta_{L_r}$ for the remaining items, and $f_r = h_1 - \delta_2$; if $L_r = 1$, put $c_r = 0$ and $f_r = h_1$. For each level, $\min\{c_r \cdot \chi : \chi \in [0,1]^{K_r}, |\chi| = \ell\}$ is the sum $c_{rk_1} + \cdots + c_{rk_\ell}$ of the ℓ smallest costs. It is attained by the incidence vector of $\{k_1,\dots,k_\ell\}$, which lies in P^r_ℓ: the term P^r_ℓ contains an integer point, the incidence vector of some set A of ℓ items with $d_r(A) \leq \bar{u}_{r\ell}$, and $d_r(\{k_1,\dots,k_\ell\}) \leq d_r(A)$ because $k_1,\dots,k_\ell$ are the ℓ lightest items. Hence $\varphi_r(\ell) = f_r + \delta_2 + \sum_{i=2}^{\ell} \delta_i = h_\ell$.

(b) is proved in full in the main text.

A.3. Size of the count chain

Let $\bar{\sigma} = \max_r |\sigma_r|$ and $L_r \le L$, so that every level lies in $\{0, \dots, L\}$. A state at stage $k-1$ is a partial sum of $k-1$ signed levels, so $|s| \le \bar{\sigma}(k-1)L$ and $|\mathcal{S}_{k-1}| \le 2\bar{\sigma}(k-1)L + 1$. The chain therefore has at most

$$\textstyle\sum_{k=1}^{m} (2\bar{\sigma}(k-1)L + 1)(L+1) = (L+1)\big(m + \bar{\sigma}L\, m(m-1)\big) = O(\bar{\sigma} m^2 L^2)$$

arcs. States on no path from $(0,0)$ to (m, n_0) can be deleted without changing the projection in Corollary 3, because in an acyclic graph a unit flow is a convex combination of source–sink paths and uses no other arc. If every σ_r is positive and $n_0 \ge 0$, partial sums never decrease, so states above n_0 lie on no path; at most $n_0 + 1$ states remain per stage and at most $m(n_0+1)(L+1)$ arcs, which is $O(m|K|L)$ for the assignment relation. The covering chain has at most $n_0 + 1$ states per stage by truncation. Since $L_r \le |K_r|$, the chain is polynomial in the size of the formulation when the coefficients σ_r are bounded, and pseudo-polynomial in $\bar{\sigma}$ otherwise.

A.4. Finding suitable orientations

This section completes the remark after Proposition 1. Reversing both endpoints of an edge exchanges meets and joins, so it preserves the sublattice property, and each edge is of one of four types: a sublattice only for equal orientations (positive), only for opposite ones (negative), for both, or for neither. The type is found by checking closure of E_{rs} under meet and join for one equal and one opposite pair of orientations, in $O(|E_{rs}|^2)$ time. Orientations exist exactly when no edge is of the fourth type and the signed graph of positive and negative edges is balanced, that is, every cycle has an even number of negative edges (Harary, *Michigan Math. J.* 2 (1953) 143–146); a breadth-first search that propagates orientations finds either an assignment or an unbalanced cycle in time linear in $|R'| + |E|$ once the edges are classified.

A.5. Verification scripts

The supplementary code contains the following scripts. They need Python 3 with NumPy and SciPy; the file `lpb.py` is a shared helper.

- `audit_theorem1.py`: not a result of this paper; it checks Theorem 2 of the companion manuscript (Anjum, Desai and Murthy, submitted to *Operations Research*), the straddle characterization of local exactness used

for the examples, on random blocks and both natural level capacities. The fractional vertices of each level polytope are enumerated exactly in rational arithmetic and compared with the straddle vertices; for every straddle vertex v, an LP confirms $(v, 1) \notin \mathrm{conv}(F_r^\circ)$. This also checks property (N), used in the last statement of Theorem 1(d).

- `audit.py`: the cycle construction of Corollary 2 for $k = 3, \ldots, 14$ in exact rational arithmetic (feasibility of the fractional point, the integer optimum, and that every profile closes r_1); the hull statement of Corollary 3 and the state bounds above, on random count relations with $\sigma_r \in \{-2, \ldots, 2\}$, for equations and coverings.
- `triangle.py`: Example 2.
- `audit_lattice.py`: Proposition 1 on random oriented sublattice couplings of 3-, 4- and 5-cycles and K_4, a mis-oriented control, the claim about Example 1, and the key step of the proof (the co-monotone coupling stays inside the pair set).
- `audit_aggregated.py`: Lemma 2(a), Theorem 2, Example 1 and Remark 1.
- `audit_midpoint.py`: Lemma 2(b), that condition (M) is equivalent to convex induced costs for every objective, and the converse in Theorem 2(c), including the instance of Remark 1.
- `audit_transfer.py`: Example 4, by enumerating all routings and computing the shortest path in the covering chain.