Multi-split Multiple-sets Implicit Convex Feasibility Seeking As Split Inverse Problems

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This paper is dedicated to Professor Simeon Reich on the occasion of his 80th birthday.

Abstract

We introduce a new class of problems, termed multi-split multiple-sets implicit convex feasibility problems (MSMS-ICFPs), define a proximity function for it and analyze its properties under certain assumptions on the involved general point-to-set operators. Additionally, we discuss its properties for a special class of point-to-set operators. For these kinds of point-to-set operators, we deploy a new iterative algorithm to solve the associated MSMS-ICFP and show that any sequence generated by it converges in norm to a solution of the MSMS-ICFP whenever a solution exists (i.e., in the consistent case). We also define a generalized proximity function for the MSMS-ICFP using the Bregman distance. Based on this function, we present a generalized simultaneous projection gradient algorithm for the split feasibility problem with multiple output sets, by examining various related results. We conclude by discussing several further special cases of the MSMS-ICFP.

Keywords: Implicit convex feasibility; Proximity function; Point-to-set operators; Split inverse problems; Bregman distance.

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1 Introduction

In this paper we define the *multi-split multiple-sets implicit convex feasibility problem* (MSMS-ICFP) and investigate it. The emphasis is on the term "implicit" which means that some of the constraints sets that describe the feasibility-seeking problem are not deterministically fixed but depend on the unknown vector itself. Put differently, these are sets that are defined via point-to-set operators.

The MSMS-ICFP studied here is an important instance of the *split inverse problem* (SIP) that deals with two different inverse problems formulated in two different spaces that are connected with each other by a bounded linear transformation. We explain below these notions and refer the reader to earlier published research works for further details.

1.1 General Background and Motivation

The *convex feasibility problem* (CFP) forms the foundation of modeling many inverse problems in various fields of mathematics and physical sciences (see, e.g., [16,19,24] and references therein). It consists of finding a common point of a finite or (countably) infinite collection of nonempty, closed, and convex subsets of a normed linear space. We consider here the finite case in the Euclidean space. Let $\{C_i\}_{i=1}^r$ be a finite collection of nonempty, closed, and convex subsets of $\mathbb{R}^n$. The CFP is:

$$\text{Find a point } x^* \in \mathbb{R}^n \text{ such that } x^* \in \bigcap_{i=1}^{r} C_i. \tag{CFP}$$

Over more than the past five decades, many researchers have extensively explored the problem (CFP) in various normed spaces and proposed algorithms to solve it (see, for instance, [2,7,16,27,28] and references therein). Many algorithms to solve the problem (CFP) belong to the class of projection methods (see, for example, [2]). However, projections onto nonempty, closed, and convex subsets of the Euclidean space or a Hilbert space can be computationally expensive or even impossible to compute explicitly. Therefore, researchers often focus on convex sets involved in the problem (CFP) for which projections are easy to compute. Recently, in [23], the problem (CFP) has been studied in the nonconvex and inconsistent settings, and they have proposed a cyclic relaxed Douglas-Rachford algorithm for it.

In general, projection algorithms are iterative methods that use projections onto nonempty, closed, and convex sets. The methods rely on the premise that when a family of sets is present in a problem then projections onto the given individual sets are easier to perform than projections onto other sets derived from them (such as intersections, image sets under some transformation, etc).

Projection algorithms compute projections onto nonempty, closed, and convex sets in various ways and, therefore, the performance mechanism of each algorithm is distinct. However, they serve to solve a variety of problems especially in optimization theory. They are capable of handling large-size problems with dimensions beyond those that can be handled by other methods available presently. Moreover, simultaneous algorithms allow parallel

computations, while sequential algorithms process one step at a time and move to the next step after having completed all the computations at the present step. We refer the reader to [10, 17] for more details on projection algorithms.

Behind the iterative algorithm scenario to solve nonlinear problems, the notion of a proximity (a.k.a. gap) function plays a crucial role in guiding the process toward a solution. This notion helps to transform various problems in nonlinear analysis into an optimization problem (see, for instance, [10,25,31] and references therein). This reformulation encourages the use of optimization techniques and algorithms to address problems in nonlinear analysis. Also, a proximity function may be used as a stopping criterion for an iterative algorithm or to find error bounds of an algorithm.

Early fundamental references for the problem (CFP) are [5, 6]. It was revisited by [13] and generalized to handle point-to-set operators, introducing the concept of *implicit convex feasibility problem* (ICFP). They demonstrated its application in image denoising within a medical imaging situation. Motivated by the above, we further generalize here the problem (ICFP) to the *multi-split multiple-sets implicit convex feasibility problem*, which is defined below.

1.2 Problem Definition

Let $\mathbb{R}^n$ and $\mathbb{R}^{m_j}$ for $j = 1, 2, \ldots, t$ be Euclidean spaces. For each $i = 1, 2, \ldots, r$, let $C_i : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a point-to-set operator such that for each $x \in \mathbb{R}^n$, the set $C_i(x)$ is nonempty, closed, and convex. For each $j = 1, 2, \ldots, t$ and $h = 1, 2, \ldots, \ell_j$, let $Q_h^j : \mathbb{R}^{m_j} \to 2^{\mathbb{R}^{m_j}}$ be a point-to-set operator such that for each $y \in \mathbb{R}^{m_j}$, the set $Q_h^j(y)$ is nonempty, closed, and convex. Let $A_j : \mathbb{R}^n \to \mathbb{R}^{m_j}$ be a bounded linear operator for each $j = 1, 2, \ldots, t$. With these given data the *multi-split multiple-sets implicit convex feasibility problem* (MSMS-ICFP) is defined as follows: find a point $x^* \in \mathbb{R}^n$ such that

$$x^* \in \bigcap_{i=1}^{r} C_i(x^*) \text{ and } A_j x^* \in \bigcap_{h=1}^{\ell_j} Q_h^j(A_j x^*), \text{ for each } j = 1, 2, \ldots, t. \tag{MSMS-ICFP}$$

In particular, if we set $t = r$, $\mathbb{R}^{m_j} = \mathbb{R}^n$, $A_j := I$, the identity operator, $Q_h^j(x) = C_j(x)$, for all $x \in \mathbb{R}^n$, $h = 1, 2, \ldots, \ell_j$, $j = 1, 2, \ldots, t$, in the above formulation, then the problem (MSMS-ICFP) reduces to the classical *implicit convex feasibility problem* (ICFP) of [13] which consists of finding a point $x^* \in \mathbb{R}^n$ such that

$$x^* \in \bigcap_{i=1}^{r} C_i(x^*). \tag{ICFP}$$

This problem was originated by [26] in the context of the cooperative wireless sensor network positioning problem, and was generalized in [13].

Note that the above problem (MSMS-ICFP) can be rewritten as follows: find a point $x^* \in \mathbb{R}^n$ such that

$$x^* \in \bigcap_{i=1}^{r} \mathrm{Fix}(C_i) \text{ and } A_j x^* \in \bigcap_{h=1}^{\ell_j} \mathrm{Fix}(Q_h^j), \text{ for each } j = 1, 2, \ldots, t, \tag{MSMS-ICFP}$$

where $\mathrm{Fix}(\cdot)$ denotes the fixed-point set of the point-to-set operator defined below.

1.3 The Split Inverse Problem and Related Work

Let X and Y be two vector spaces, and $A : X \to Y$ be a bounded linear operator. In addition, two inverse problems are involved. The first one is IP_1, formulated in X and the second one is IP_2, formulated in Y. With these given data, the *split inverse problem* (SIP), introduced in [15, Section 2] is defined as follows:

$$\begin{aligned} &\text{Find a point } \; x^* \in X \text{ that solves } \mathrm{IP}_1 \\ &\qquad\qquad \text{such that} \\ &\text{the point } y^* = Ax^* \in Y \text{ solves } \mathrm{IP}_2. \end{aligned} \tag{SIP}$$

SIPs are quite general because they enable one to obtain various split problems by making different choices of IP_1 and IP_2. Due to their significant applications in, e.g., the intensity-modulated radiation therapy treatment problem [1, 11] and in economic theory [32], various types of SIPs have recently attracted considerable attention. One notable example is the split variational inequality problem (SVIP) [15]. For comprehensive discussions of different types of split problems, see [20–22, 33, 35, 36, 41, 42] and references therein. The problem (MSMS-ICFP) studied in this paper is an important instance of a SIP.

Current interest and activity in the field of split problems is further evidenced by several recent relevant papers, including [26, 29, 38, 43, 44].

1.4 Our Contribution

The problem (MSMS-ICFP) provides a unified structure for feasibility problems involving implicit constraint sets in the domain space, together with multiple split implicit constraint sets in possibly different image spaces. Our approach combines the notions of proximity functions, projection methods, and Bregman distances. The main contributions to this work are as follows.

1. We introduce a new feasibility-seeking problem called the multi-split multiple-sets implicit convex feasibility problem (MSMS-ICFP). In this problem, an unknown point is required to satisfy a finite family of implicit constraint sets in the domain space, while its images under bounded linear operators are required to belong to prescribed families of implicit constraint sets in the relevant image spaces. The problem (MSMS-ICFP) generalizes the implicit convex feasibility problem (ICFP).

2. We employ a proximity function for the problem (MSMS-ICFP) and establish its continuity under appropriate assumptions on the point-to-set operators.

3. For a special class of point-to-set operators involving translations, scalings, and unitary transformations of fixed nonempty, closed, and convex sets, we obtain an explicit formulation of the proximity function. In this setting, we show that the proximity function is convex, continuously differentiable, and its gradient is Lipschitz continuous.

4. We propose a simultaneous projection gradient algorithm (SPGA) for solving the problem (MSMS-ICFP) and prove that the sequence generated by it converges in norm to a solution of the problem (MSMS-ICFP), whenever one exists.

5. We introduce a generalized proximity function for the problem (MSMS-ICFP) by replacing Euclidean distances and orthogonal projections by Bregman distances and Bregman projections, respectively. Since the domain and image constraints are defined in different Euclidean spaces, we consider a Bregman function in the domain space and other Bregman functions in the corresponding image spaces.

6. The split feasibility problem with multiple output sets is a special case of the problem (MSMS-ICFP) with fixed constraint sets. We derive the differentiability and Lipschitz continuity of the gradient of the generalized proximity function for the problem under suitable assumptions. One of these is the Lipschitz continuity assumptions on the Hessian-weighted Bregman projection residual operators. We show that this assumption is more appropriate than imposing global boundedness of the projection residuals. In particular, we give an example to show that in the Euclidean setting for arbitrary nonempty, closed, and convex constraint sets the proposed assumptions hold, even when the associated residuals are unbounded.

7. Based on the generalized proximity function, we establish a generalized simultaneous projection gradient algorithm (GSPGA) for the split feasibility problem with multiple output sets and prove that the sequence generated by the algorithm converges in norm to a solution of the split feasibility problem with multiple output sets, whenever one exists.

8. We establish various special cases of the problem (MSMS-ICFP) such as the split common fixed point problem, the fixed point problem, the multiple-sets split feasibility problem, the split feasibility problem, the convex feasibility problem, and the split common minimization problem.

The paper is organized as follows. In Section 2 we review some existing definitions and results. In Section 3 we introduce a proximity function for the problem (MSMS-ICFP) and examine its properties for general point-to-set operators. We also discuss the properties of the proximity function for a special class of point-to-set operators. Additionally, we present a new iterative algorithm to solve the problem (MSMS-ICFP) and prove that the sequence generated by the algorithm converges to a solution, if one exists. We also define a generalized proximity function for the MSMS-ICFP, and based on it, present a generalized simultaneous projection gradient algorithm (GSPGA) for the split feasibility problem with multiple output sets, by examining various related results. In Section 4 we discuss various special cases of the problem (MSMS-ICFP). In Section 5 we draw a conclusion of our work.

2 Preliminaries

In this section, we recall some basic definitions and results that will be used later in the paper. Throughout this paper, I denotes the identity operator on the Euclidean space whose dimension is determined by the context. Additionally, $\|\cdot\|$ denotes the Euclidean norm induced by the inner product $\langle\cdot,\cdot\rangle$ in the Euclidean space.

Definition 2.1 *[10, p. 28] A function $f:\mathbb{R}^n\to\mathbb{R}$ is said to be a proximity function for the problem (MSMS-ICFP) if it satisfies the following properties:*

(P_1) $f(x)\geq 0$, *for all* $x\in\mathbb{R}^n$,

(P_2) $f(x^*)=0$, $x^*\in\mathbb{R}^n$ *if and only if* x^* *is a solution of the problem* (MSMS-ICFP).

Definition 2.2 *[4] Let C be a nonempty subset of $\mathbb{R}^n$. An operator $T:C\to\mathbb{R}^n$ is said to be Lipschitz continuous with constant $L>0$ if*

$$\|T(x)-T(y)\|\leq L\|x-y\|,\ \text{for all } x,y\in C. \tag{1}$$

Lemma 2.1 *[30, Theorem 2.4] A differentiable function $f:\mathbb{R}^n\to\mathbb{R}$ is σ-strongly convex for some $\sigma>0$ if and only if its gradient ∇f is σ-strongly monotone, that is,*

$$\langle\nabla f(y)-\nabla f(x),y-x\rangle\geq\sigma\|y-x\|^2,\ \ \textit{for all } x,y\in\mathbb{R}^n. \tag{2}$$

Definition 2.3 *Let C be a nonempty subset of $\mathbb{R}^n$ and let $S:C\to 2^C$ be a point-to-set operator. Then the set of all fixed points of S is denoted by $Fix(S)$ and defined as follows:*

$$Fix(S):=\{x\in C\mid x\in S(x)\}. \tag{3}$$

Definition 2.4 *[4, pp. 50–51] Let C be a nonempty subset of $\mathbb{R}^n$. The distance of a point $x\in\mathbb{R}^n$ to a set C is denoted by $dist(x,C)$, i.e.,*

$$dist(x,C)=\inf_{y\in C}\|x-y\| \tag{4}$$

and the set of points at which distance is attended is called the nearest point projection of x onto C is defined by

$$P_C(x):=\arg\min_{y\in C}\|x-y\|. \tag{5}$$

Remark 2.1 *1. [4, Theorem 3.16 (Projection theorem)] If C is a nonempty, closed, and convex subset of $\mathbb{R}^n$, then for each point $x\in\mathbb{R}^n$, the projection onto C exists and is unique. Furthermore, for every $x,z\in\mathbb{R}^n$*

$$z=P_C(x)\Leftrightarrow\langle x-z,y-z\rangle\leq 0,\ \text{for all } y\in C. \tag{6}$$

2. *Using the characterization of projection operator mentioned in (6), one can verify that it is nonexpansive. That is,*

$$\|P_C(x) - P_C(y)\| \leq \|x - y\|, \text{ for all } x, y \in \mathbb{R}^n. \tag{7}$$

3. *From the definition of the projection operator, it is clear that* $Fix(P_C) = \{x | x = P_C(x)\} = \{x | x \in C\} = C$.

Lemma 2.2 *[37, Proof of Lemma 3.2] Let* C *be a nonempty, closed, and convex subset of* $\mathbb{R}^n$*, and let* $\nu : \mathbb{R}^n \to \mathbb{R}^n$ *be an operator. Let* $\Omega : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ *be a point-to-set operator such that* $\Omega(x) = \nu(x) + C = \{\nu(x) + c | c \in C\}$ *for every* $x \in \mathbb{R}^n$*. Then for every* $x, z \in \mathbb{R}^n$

$$P_{\Omega(x)}(z) = \nu(x) + P_C(z - \nu(x)). \tag{8}$$

Lemma 2.3 *[13, Lemma 2.1] Let* C *be a nonempty, closed, and convex subset of* $\mathbb{R}^n$*. Then for a given matrix* A*, a positive diagonal matrix* $D = \alpha I$*,* $\alpha > 0$*, and a fixed unitary matrix* U *(that is,* $UU^T = U^TU = I$*), the following is true*

$$\begin{aligned} P_{DU(C)+Ax}(y) &= DUP_C(D^{-1}U^T(y - Ax)) + Ax \\ &= \alpha UP_C\Big(\frac{1}{\alpha}U^T(y - Ax)\Big) + Ax, \text{ for all } x, y \in \mathbb{R}^n, \end{aligned} \tag{9}$$

where the superscript T *stands for the transpose and* $U(C) := \{Ux \mid x \in C\}$.

We deduce the following corollary from [18, Theorem 3.4]. Note that $\arg\min_{x\in\mathbb{R}^n} f(x)$ is the set of all minimizers of f.

Corollary 2.1 *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a convex function with Lipschitz continuous gradient with constant* $L > 0$*. Assume that the solution set* G *of the problem* $\min_{x\in\mathbb{R}^n} f(x)$ *is nonempty. Let* $x^0 \in \mathbb{R}^n$ *be arbitrary and* $\lambda \in (0, 2/L)$*. Then the sequence* $\{x^k\}_{k=0}^{\infty}$ *generated by the gradient method* $x^{k+1} := x^k - \lambda\nabla f(x^k)$*, converges in norm to a point in* G.

Proof. We know that $prox_{\lambda f}(x) = \arg\min_{y\in\mathbb{R}^n}\{f(y) + (1/2\lambda)\|x - y\|^2\}$ for $x \in \mathbb{R}^n$. If $f = 0$ then $prox_{\lambda f}(x) = x$ for $x \in \mathbb{R}^n$. Under our assumptions, the proof directly follows from [18, Theorem 3.4 (i)] by substituting $\mathscr{H} = \mathbb{R}^n$, $\lambda_k = 1, \gamma_k = \lambda, a_k = 0, b_k = 0$ for all $k \geq 0$, $\beta = 1/L$, $f_1 = 0$, $f_2 = f$, and using the fact that $prox_{\lambda f_1}(x) = x$ for $x \in \mathbb{R}^n$. ■

Remark 2.2 *It is worth noting that Corollary 2.1 also can be deduced from [4, Corollary 27.10] under an appropriate choice of parameters in the Euclidean space setting.*

It is well-known that for a given matrix $A \in \mathbb{R}^{m\times n}$, its 2-norm is

$$\|A\|_2 = \sup\{x^TAy | x \in \mathbb{R}^m, y \in \mathbb{R}^n \text{ with } \|x\| = 1 = \|y\|\}. \tag{10}$$

Furthermore, it satisfies

$$\|A^TA\|_2 = \|AA^T\|_2 = \|A\|_2^2 \text{ and } \|A\|_2 = \|A^T\|_2. \tag{11}$$

3 Main Results: Proximity Functions and Algorithms

3.1 The Proximity Function and its Characterization

Let $C_i : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a point-to-set operator for $i = 1, 2, \ldots, r$. Let, for each $x \in \mathbb{R}^n$, the set $C_i(x)$ be nonempty, closed, and convex. For each $j = 1, 2, \ldots, t$ and $h = 1, 2, \ldots, \ell_j$, let $Q_h^j : \mathbb{R}^{m_j} \to 2^{\mathbb{R}^{m_j}}$ be a finite family of point-to-set operators. Each set $Q_h^j(y)$ is nonempty, closed, and convex for every $y \in \mathbb{R}^{m_j}$. Let $A_j : \mathbb{R}^n \to \mathbb{R}^{m_j}$ be a bounded linear operator for each $j = 1, 2, \ldots, t$.

Define a function $g : \mathbb{R}^n \to \mathbb{R}$ by

$$g(x) := \frac{1}{2}\sum_{i=1}^{r} \alpha_i \|x - P_{C_i(x)}(x)\|^2 + \frac{1}{2}\sum_{j=1}^{t}\sum_{h=1}^{\ell_j} \beta_h^j \|A_j x - P_{Q_h^j(A_j x)}(A_j x)\|^2, \ x \in \mathbb{R}^n, \tag{12}$$

where α_i and β_h^j are positive constants for all h, i, j.

From Remark 2.1(1), it is evident that the function g is well-defined and is a real-valued function on $\mathbb{R}^n$.

Proposition 3.1 *A point $x^* \in \mathbb{R}^n$ is a solution of the problem (MSMS-ICFP) if and only if it solves the equation $g(x) = 0$.*

Proof. Let $x^* \in \mathbb{R}^n$ be a solution of the problem (MSMS-ICFP). Then using the projection characterization (Remark 2.1(3)) it is straightforward to verify that

$$x^* = P_{C_i(x^*)}(x^*) \text{ and } A_j x^* = P_{Q_h^j(A_j x^*)}(A_j x^*), \text{ for all } h, i, j. \tag{13}$$

This implies that $g(x^*) = 0$.

Conversely, suppose $x^* \in \mathbb{R}^n$ solves the equation $g(x) = 0$. This implies that

$$\frac{1}{2}\sum_{i=1}^{r} \alpha_i \|x^* - P_{C_i(x^*)}(x^*)\|^2 + \frac{1}{2}\sum_{j=1}^{t}\sum_{h=1}^{\ell_j} \beta_h^j \|A_j x^* - P_{Q_h^j(A_j x^*)}(A_j x^*)\|^2 = 0 \tag{14}$$

yielding

$$\frac{1}{2}\sum_{i=1}^{r} \alpha_i \|x^* - P_{C_i(x^*)}(x^*)\|^2 = 0 \text{ and } \frac{1}{2}\sum_{j=1}^{t}\sum_{h=1}^{\ell_j} \beta_h^j \|A_j x^* - P_{Q_h^j(A_j x^*)}(A_j x^*)\|^2 = 0. \tag{15}$$

Consequently,

$$\begin{aligned} &\alpha_i \|x^* - P_{C_i(x^*)}(x^*)\|^2 = 0, \text{ for all } i = 1, 2, \ldots, r, \text{ and} \\ &\sum_{h=1}^{\ell_j} \beta_h^j \|A_j x^* - P_{Q_h^j(A_j x^*)}(A_j x^*)\|^2 = 0, \text{ for all } j = 1, 2, \ldots, t. \end{aligned} \tag{16}$$

Therefore,

$$x^* \in \bigcap_{i=1}^{r} C_i(x^*) \text{ and } A_j x^* \in \bigcap_{h=1}^{\ell_j} Q_h^j(A_j x^*), \text{ for each } j = 1, 2, \ldots, t. \tag{17}$$

Thus, $x^* \in \mathbb{R}^n$ is a solution of the problem (MSMS-ICFP). ■

Now, we are in the position to present the following result.

Proposition 3.2 *The function $g : \mathbb{R}^n \to \mathbb{R}$, defined above in (12), is a proximity function for the problem (MSMS-ICFP).*

Proof. By definition $g(x)$ is non-negative for all $x \in \mathbb{R}^n$. From Proposition 3.1, it is evident that g satisfies property (P_2). Thus, all the conditions in Definition 2.1 are satisfied. ■

Corollary 3.1 *The problem (MSMS-ICFP) is equivalent to the unconstrained minimization problem:*

$$\min_{x \in \mathbb{R}^n} g(x). \tag{18}$$

Before proceeding to prove the continuity of the proximity function g defined in (12), let us establish the following Proposition.

Proposition 3.3 *Let $C : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a point-to-set operator such that $C(x)$ is nonempty, closed, and convex for each $x \in \mathbb{R}^n$. Assume that there exists a constant $\eta > 0$ such that*

$$\|P_{C(x)}(z) - P_{C(y)}(z)\| \leq \eta \|x - y\|, \text{ for all } x, y, z \in \mathbb{R}^n. \tag{19}$$

Then the function $\pi(x) := P_{C(x)}(x)$, which maps each $x \in \mathbb{R}^n$ to its projection onto the set $C(x)$, is continuous on $\mathbb{R}^n$.

Proof. Let $\{x^k\}_{k=0}^{\infty}$ be a sequence in $\mathbb{R}^n$ such that $x^k \to x \in \mathbb{R}^n$ as $k \to \infty$. Then

$$\begin{aligned}
\|\pi(x^k) - \pi(x)\| &= \|P_{C(x^k)}(x^k) - P_{C(x)}(x)\| \\
&= \|P_{C(x^k)}(x^k) - P_{C(x^k)}(x) + P_{C(x^k)}(x) - P_{C(x)}(x)\| \\
&\leq \|P_{C(x^k)}(x^k) - P_{C(x^k)}(x)\| + \|P_{C(x^k)}(x) - P_{C(x)}(x)\| \\
&\leq \|x^k - x\| + \eta \|x^k - x\| \\
&\leq (1 + \eta)\|x^k - x\| \to 0 \text{ as } k \to \infty.
\end{aligned} \tag{20}$$

Therefore, $\pi(x^k) \to \pi(x)$ as $k \to \infty$ and the proof is complete. ■

Remark 3.1 *Let $C : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a point-to-set operator such that $C(x)$ is nonempty, closed, and convex for each $x \in \mathbb{R}^n$. Assume that $C(x) = f(x) + \Omega$, where Ω is a nonempty, closed, and convex subset of $\mathbb{R}^n$, and f is a Lipschitz continuous function on $\mathbb{R}^n$ into itself with Lipschitz constant L. Then the inequality (19) holds with the constant L. For more details see, e.g., [37, Lemma 3.2].*

Finally, we prove the continuity of the proximity function g defined in (12) under certain assumptions on the point-to-set operators involved.

Proposition 3.4 *Let, for each $i = 1, 2, \ldots, r$, $C_i : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a point-to-set operator such that $C_i(x)$ is nonempty, closed, and convex for each $x \in \mathbb{R}^n$. Let, for each $j = 1, 2, \ldots, t$ and $h = 1, 2, \ldots, \ell_j$, $Q_h^j : \mathbb{R}^{m_j} \to 2^{\mathbb{R}^{m_j}}$ be a finite family of point-to-set operators of $\mathbb{R}^{m_j}$ such that $Q_h^j(u)$ is nonempty, closed, and convex for each $u \in \mathbb{R}^{m_j}$. Let $A_j : \mathbb{R}^n \to \mathbb{R}^{m_j}$ be a bounded linear operator for each $j = 1, 2, \ldots, t$. Assume that there exist positive constants $\{\eta_i\}_{i=1}^r$ and $\{\delta_h^j\}_{h=1,j=1}^{\ell_j,t}$ such that*

$$\|P_{C_i(x)}(z) - P_{C_i(y)}(z)\| \leq \eta_i \|x - y\|, \text{ for all } x, y, z \in \mathbb{R}^n, i = 1, 2, \ldots, r, \tag{21}$$

and

$$\|P_{Q_h^j(A_j x)}(A_j z) - P_{Q_h^j(A_j y)}(A_j z)\| \leq \delta_h^j \|A_j x - A_j y\|, \text{ for all } x, y, z \in \mathbb{R}^n, h = 1, 2, \ldots, \ell_j, j = 1, 2, \ldots, t. \tag{22}$$

Then the function g is continuous on $\mathbb{R}^n$.

Proof. Using the assumptions and Proposition 3.3, it follows that for each i, the function $\|x - P_{C_i(x)}(x)\|^2$ is continuous for any $x \in \mathbb{R}^n$. Similarly, for each j, h, the function $\|A_j x - P_{Q_h^j(A_j x)}(A_j x)\|^2$ is continuous for any $x \in \mathbb{R}^n$, since each A_j is a bounded linear operator. From the well-known fact that a finite linear combination of continuous functions is continuous, it follows that the function g of (12) is continuous for all $x \in \mathbb{R}^n$. ■

3.2 Algorithms for the Multi-Split Multiple-Sets Implicit Convex Feasibility Problem

For each $x \in \mathbb{R}^n$, let the function g be of the form:

$$g(x) := u(x) + v(x), \tag{23}$$

where the summands are

$$u(x) := \frac{1}{2} \sum_{i=1}^{r} \alpha_i \|x - P_{C_i(x)}(x)\|^2 \tag{24}$$

and

$$v(x) := \frac{1}{2} \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j \|A_j x - P_{Q_h^j(A_j x)}(A_j x)\|^2, \tag{25}$$

and $\{\alpha_i\}_{i=1}^r$ and $\{\beta_h^j\}_{h=1,j=1}^{\ell_j,t}$ are positive constants.

From this point onward we assume that the point-to-set operators are of the following form:

$$\begin{cases} C_i(x) := \gamma_i U_i(\Omega_i) + M_i x \\ Q_h^j(A_j x) := \nu_h^j W_h^j(\Theta_h^j) + N_h^j A_j x, \end{cases} \tag{A1}$$

where $x \in \mathbb{R}^n$, $M_i \in \mathbb{R}^{n\times n}$ and $N_h^j \in \mathbb{R}^{m_j\times m_j}$ are matrices, $U_i \in \mathbb{R}^{n\times n}, W_h^j \in \mathbb{R}^{m_j\times m_j}$ are unitary matrices, Ω_i and Θ_h^j are nonempty, closed, and convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^{m_j}$, respectively, $\gamma_i > 0$ and $\nu_h^j > 0$ are fixed constants for all h, j, i. In this section of the paper, we assume that (A1) holds. These operators generalize the point-to-set operators that are used in Lemma 2.3.

Remark 3.2 *1. Using Lemma 2.3, for $x \in \mathbb{R}^n$, it is straightforward that*

$$u(x) = \frac{1}{2}\sum_{i=1}^{r} \alpha_i \gamma_i^2 \|(I - P_{\Omega_i})(K_i x)\|^2, \tag{26}$$

where $K_i := \frac{1}{\gamma_i} U_i^T (I - M_i)$ for $i = 1, 2, \ldots, r$.

2. From [13, Proposition 3.1] it is true that the function u is convex, continuously differentiable on $\mathbb{R}^n$, and its gradient is

$$\nabla u(x) = \sum_{i=1}^{r} \alpha_i \gamma_i^2 K_i^T (I - P_{\Omega_i})(K_i x),\ x \in \mathbb{R}^n. \tag{27}$$

Furthermore, ∇u is Lipschitz continuous with constant

$$L_{\nabla u} := \sum_{i=1}^{r} \alpha_i \|I - M_i\|_2^2. \tag{28}$$

Indeed, using the fact that the projection operator is firmly nonexpansive, the definition of matrix norm, and $\|K_i^T K_i\|_2 = (1/\gamma_i^2)\|I - M_i\|_2^2$, for any $x, y \in \mathbb{R}^n$ we obtain

$$\begin{aligned}
&\|\nabla u(x) - \nabla u(y)\| \\
&\quad = \Big\| \sum_{i=1}^{r} \alpha_i \gamma_i^2 K_i^T (I - P_{\Omega_i})(K_i x) - \sum_{i=1}^{r} \alpha_i \gamma_i^2 K_i^T (I - P_{\Omega_i})(K_i y) \Big\| \\
&\quad \le \sum_{i=1}^{r} \alpha_i \gamma_i^2 \|K_i^T\|_2 \|(I - P_{\Omega_i})(K_i x) - (I - P_{\Omega_i})(K_i y)\| \\
&\quad \le \sum_{i=1}^{r} \alpha_i \gamma_i^2 \|K_i^T\|_2 \|K_i x - K_i y\| \\
&\quad \le \sum_{i=1}^{r} \alpha_i \gamma_i^2 \|K_i^T\|_2 \|K_i\|_2 \|x - y\| \\
&\quad = \sum_{i=1}^{r} \alpha_i \gamma_i^2 \|K_i\|_2^2 \|x - y\| \\
&\quad = \sum_{i=1}^{r} \alpha_i \gamma_i^2 \|K_i^T K_i\|_2 \|x - y\|
\end{aligned}$$

$$= \sum_{i=1}^{r} \alpha_i \gamma_i^2 \Big(\frac{1}{\gamma_i^2} \|I - M_i\|_2^2 \Big) \|x - y\|$$
$$= \sum_{i=1}^{r} \alpha_i \|I - M_i\|_2^2 \|x - y\|. \tag{29}$$

3. *The assumption (A1) satisfies the continuity assumption on (21) and (22). Indeed, since $C_i(x) := \gamma_i U_i(\Omega_i) + M_i x$ is a translation of the fixed closed convex set $\gamma_i U_i(\Omega_i)$, the nonexpansiveness of the projection operator and Lemma 2.2 imply that*

$$\|P_{C_i(x)}(z) - P_{C_i(y)}(z)\| \leq 2\|M_i\|_2 \|x - y\|, \text{ for all } x, y, z \in \mathbb{R}^n, i = 1, 2, \ldots, r. \tag{30}$$

Hence, the condition (21) holds with $\eta_i = 2\|M_i\|_2$. Similarly, since $Q_h^j(A_j x) := \nu_h^j W_h^j(\Theta_h^j) + N_h^j A_j x$, the nonexpansiveness of the projection operator and Lemma 2.2 imply that

$$\|P_{Q_h^j(A_j x)}(A_j z) - P_{Q_h^j(A_j y)}(A_j z)\| \leq 2 \ \|N_h^j\|_2 \ \|A_j x - A_j y\|, \text{for all } x, y, z \in \mathbb{R}^n, \ h = 1, 2, \ldots, \ell_j, j = 1, 2, \ldots, t. \tag{31}$$

Hence, the condition (22) holds with $\delta_h^j = 2\|N_h^j\|_2$.

4. *In (A1), we assumed that Q_h^j is specified only for Range$(A_j) = \{A_j x | x \in \mathbb{R}^n\}$, i.e.,*

$$Q_h^j(A_j x) = \nu_h^j W_h^j(\Theta_h^j) + N_h^j A_j x, \ x \in \mathbb{R}^n.$$

This is weaker than assuming that

$$Q_h^j(u) = \nu_h^j W_h^j(\Theta_h^j) + N_h^j u, \ u \in \mathbb{R}^{m_j},$$

which defines Q_h^j on the whole space $\mathbb{R}^{m_j}$. However, when Range$(A_j) = \mathbb{R}^{m_j}$, i.e., when A_j is surjective, both assumptions are equivalent.

Proposition 3.5 *The function v is convex, continuously differentiable on $\mathbb{R}^n$ and its gradient is given by the formula*

$$\nabla v(x) = \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j (\nu_h^j)^2 (L_h^j)^T (I - P_{\Theta_h^j})(L_h^j x), \ x \in \mathbb{R}^n. \tag{32}$$

Furthermore, ∇v is Lipschitz continuous with constant

$$L_{\nabla v} := \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j \|(I - N_h^j) A_j\|_2^2. \tag{33}$$

Proof. For any $x \in \mathbb{R}^n$, using (A1) and Lemma 2.3, we obtain,

$$\begin{aligned}
v(x) &= \frac{1}{2}\sum_{j=1}^{t}\sum_{h=1}^{\ell_j}\beta_h^j\|A_jx - P_{Q_h^j(A_jx)}(A_jx)\|^2 \\
&= \frac{1}{2}\sum_{j=1}^{t}\sum_{h=1}^{\ell_j}\beta_h^j\left\|A_jx - P_{(\nu_h^jW_h^j(\Theta_h^j)+N_h^jA_jx)}(A_jx)\right\|^2 \\
&= \frac{1}{2}\sum_{j=1}^{t}\sum_{h=1}^{\ell_j}\beta_h^j\left\|A_jx - \nu_h^jW_h^jP_{\Theta_h^j}\Big(\frac{1}{\nu_h^j}(W_h^j)^T(A_jx - N_h^jA_jx)\Big) - N_h^jA_jx\right\|^2 \\
&= \frac{1}{2}\sum_{j=1}^{t}\sum_{h=1}^{\ell_j}\beta_h^j\left\|\nu_h^jW_h^j\Big(\frac{1}{\nu_h^j}(W_h^j)^T(A_jx - N_h^jA_jx) - P_{\Theta_h^j}\Big(\frac{1}{\nu_h^j}(W_h^j)^T(A_jx - N_h^jA_jx)\Big)\Big)\right\|^2 \\
&= \frac{1}{2}\sum_{j=1}^{t}\sum_{h=1}^{\ell_j}\beta_h^j(\nu_h^j)^2\left\|\Big(I - P_{\Theta_h^j}\Big)\Big(\frac{1}{\nu_h^j}(W_h^j)^T(A_jx - N_h^jA_jx)\Big)\right\|^2 \\
&= \frac{1}{2}\sum_{j=1}^{t}\sum_{h=1}^{\ell_j}\beta_h^j(\nu_h^j)^2\|(I - P_{\Theta_h^j})(L_h^jx)\|^2, \qquad (34)
\end{aligned}$$

where

$$L_h^j := \frac{1}{\nu_h^j}(W_h^j)^T(I - N_h^j)A_j. \qquad (35)$$

Clearly, $L_h^j \in \mathbb{R}^{m_j\times n}$. Using the definition of matrix norm, we get

$$\|L_h^jx - L_h^jy\| \le \|L_h^j\|_2\|x - y\|, \qquad (36)$$

for $x, y \in \mathbb{R}^n$.

The convexity of v follows from (34), see for instance [34, pp. 367–368]. Using the fact of [13, Equation (3.15), p. 617], we obtain

$$\nabla v(x) = \sum_{j=1}^{t}\sum_{h=1}^{\ell_j}\beta_h^j(\nu_h^j)^2(L_h^j)^T(I - P_{\Theta_h^j})(L_h^jx). \qquad (37)$$

Now,

$$\begin{aligned}
(L_h^j)^TL_h^j &= \frac{1}{(\nu_h^j)^2}((W_h^j)^T(I - N_h^j)A_j)^T(W_h^j)^T(I - N_h^j)A_j \\
&= \frac{1}{(\nu_h^j)^2}((I - N_h^j)A_j)^T(I - N_h^j)A_j. \qquad (38)
\end{aligned}$$

Again,

$$\|(L_h^j)^TL_h^j\|_2 = \tfrac{1}{(\nu_h^j)^2}\|((I - N_h^j)A_j)^T(I - N_h^j)A_j\|_2 = \Big(\tfrac{1}{\nu_h^j}\|(I - N_h^j)A_j\|_2\Big)^2. \qquad (39)$$

Finally, for any $x, y \in \mathbb{R}^n$, using the fact that the projection operator is firmly nonexpansive, the definition of matrix norm, (36), and (39), we obtain

$$\begin{aligned}
&\|\nabla v(x) - \nabla v(y)\| \\
&\quad = \Big\| \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j (\nu_h^j)^2 (L_h^j)^T (I - P_{\Theta_h^j})(L_h^j x) - \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j (\nu_h^j)^2 (L_h^j)^T (I - P_{\Theta_h^j})(L_h^j y) \Big\| \\
&\quad = \Big\| \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j (\nu_h^j)^2 (L_h^j)^T ((I - P_{\Theta_h^j})(L_h^j x) - (I - P_{\Theta_h^j})(L_h^j y)) \Big\| \\
&\quad \leq \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j (\nu_h^j)^2 \|(L_h^j)^T\|_2 \|L_h^j x - L_h^j y\| \\
&\quad \leq \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j (\nu_h^j)^2 \|(L_h^j)^T\|_2 \|L_h^j\|_2 \|x - y\| \\
&\quad = \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j (\nu_h^j)^2 \|L_h^j\|_2^2 \|x - y\| \\
&\quad = \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j (\nu_h^j)^2 \|(L_h^j)^T L_h^j\|_2 \|x - y\| \\
&\quad = \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j (\nu_h^j)^2 \Big(\frac{1}{(\nu_h^j)^2} \|(I - N_h^j) A_j\|_2^2 \Big) \|x - y\| \\
&\quad = \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j \|(I - N_h^j) A_j\|_2^2 \|x - y\|. \qquad (40)
\end{aligned}$$

This implies that ∇v is $\sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j \|(I - N_h^j) A_j\|_2^2$-Lipschitz continuous. ■

Proposition 3.6 *The function g of (12) is convex and continuously differentiable on $\mathbb{R}^n$ and its gradient is*

$$\nabla g(x) = \sum_{i=1}^{r} \alpha_i \gamma_i^2 K_i^T (I - P_{\Omega_i})(K_i x) + \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j (\nu_h^j)^2 (L_h^j)^T (I - P_{\Theta_h^j})(L_h^j x), \; x \in \mathbb{R}^n, \quad (41)$$

where $\{\alpha_i\}_{i=1}^r, \{\gamma_i\}_{i=1}^r, \{\beta_h^j\}_{h=1,j=1}^{\ell_j,t}, \{\nu_h^j\}_{h=1,j=1}^{\ell_j,t}$ are positive constants. Furthermore, ∇g is $L_{\nabla g} := \sum_{i=1}^{r} \alpha_i \|I - M_i\|_2^2 + \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j \|(I - N_h^j) A_j\|_2^2$-Lipschitz continuous. That is,

$$\|\nabla g(x) - \nabla g(y)\| \leq L_{\nabla g} \|x - y\|, \text{ for all } x, y \in \mathbb{R}^n. \qquad (42)$$

Proof. The function g is convex and continuously differentiable since the functions u and v are both convex and continuously differentiable on $\mathbb{R}^n$ (see Remark 3.2 and Proposition 3.5).

Since for any $x \in \mathbb{R}^n$, $g(x) = u(x) + v(x)$, it follows that $\nabla g(x) = \nabla u(x) + \nabla v(x)$. Thereby, for any $x, y \in \mathbb{R}^n$ we have

$$\begin{aligned} \|\nabla g(x) - \nabla g(y)\| &= \|\nabla u(x) + \nabla v(x) - \nabla u(y) - \nabla v(y)\| \\ &\leq \|\nabla u(x) - \nabla u(y)\| + \|\nabla v(x) - \nabla v(y)\|. \end{aligned} \tag{43}$$

Since ∇u is $\sum_{i=1}^{r} \alpha_i \|I - M_i\|_2^2$-Lipschitz continuous (see Remark 3.2) and ∇v is $\sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j \|(I - N_h^j)A_j\|_2^2$-Lipschitz continuous (see Proposition 3.5), from inequality (43) we obtain

$$\begin{aligned} &\| \ \nabla g(x) - \nabla g(y)\| \\ &\leq \|\nabla u(x) - \nabla u(y)\| + \|\nabla v(x) - \nabla v(y)\| \\ &\leq \Big(\sum_{i=1}^{r} \alpha_i \|I - M_i\|_2^2 + \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j \|(I - N_h^j)A_j\|_2^2 \Big) \|x - y\|, \text{ for all } x, y \in \mathbb{R}^n. \end{aligned} \tag{44}$$

Therefore, ∇g is $\sum_{i=1}^{r} \alpha_i \|I - M_i\|_2^2 + \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j \|(I - N_h^j)A_j\|_2^2$-Lipschitz continuous. ■

3.2.1 The simultaneous algorithm and its convergence

In this subsection, we introduce a new simultaneous (parallel) projection algorithm to solve the problem (MSMS-ICFP) for point-to-set operators of the form as in (A1).

Algorithm 1 A Simultaneous Projection Gradient Algorithm (SPGA) for MSMS-ICFP

Input: $x^0 \in \mathbb{R}^n$.

Compute $L_{\nabla g} = \sum_{i=1}^{r} \alpha_i \|I - M_i\|_2^2 + \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j \|(I - N_h^j)A_j\|_2^2$.

for $i = 1, 2, \ldots, r$ **do**

Compute

$$K_i = \frac{1}{\gamma_i} U_i^T (I - M_i). \tag{45}$$

for $j = 1, 2, \ldots, t$ **do**

for $h = 1, 2, \ldots, \ell_j$ **do**

Compute

$$L_h^j = \frac{1}{\nu_h^j} (W_h^j)^T (I - N_h^j) A_j. \tag{46}$$

Choose $\lambda \in (0, 2/L_{\nabla g})$.
for $k = 0, 1, 2, \ldots$ **do**
    Compute

$$x^{k+1} := x^k - \lambda \sum_{i=1}^{r} \alpha_i \gamma_i^2 K_i^T (I - P_{\Omega_i})(K_i x^k) - \lambda \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j (\nu_h^j)^2 (L_h^j)^T (I - P_{\Theta_h^j})(L_h^j x^k). \tag{47}$$

Remark 3.3 *In Algorithm 1, we use a constant step-size and prove the convergence of the sequence $\{x^k\}_{k=0}^{\infty}$ generated by it in Theorem 3.1 below. However, our proof of convergence carries over also with an appropriate choice of variable step-sizes.*

Theorem 3.1 *Denote the solution set of the problem (MSMS-ICFP) by Θ and assume that it is nonempty. Then any sequence $\{x^k\}_{k=0}^{\infty}$ generated by Algorithm 1 converges in norm to a point in Θ.*

Proof. Since Θ is nonempty, the problem (MSMS-ICFP) is consistent. Therefore, by Proposition 3.2 and Corollary 3.1, the solution set of (MSMS-ICFP) coincides with the global minimizers of the unconstrained minimization problem (18) and the minimum value is zero. Moreover, by Proposition 3.6, the function g of (12) is convex, continuously differentiable, and its gradient is $L_{\nabla g}$-Lipschitz continuous. Since the step-size $\lambda \in (0, 2/L_{\nabla g})$, all the assumptions of Corollary 2.1 are satisfied. By the expression in (47), our Algorithm 1 is a standard gradient descent algorithm for the problem (MSMS-ICFP). Therefore, by applying Corollary 2.1, we conclude that the sequence $\{x^k\}_{k=0}^{\infty}$ generated by Algorithm 1 converges in norm to a point $y^* \in \arg\min_{x \in \mathbb{R}^n} g(x)$. Since the set of all global minimizers of g coincides with Θ, $y^* \in \Theta$. Thus, the sequence $\{x^k\}_{k=0}^{\infty}$ converges in norm to a solution of the problem (MSMS-ICFP). ■

Remark 3.4 *The above analysis holds for a special type of point-to-set operators. Therefore, it is natural to ask whether it can be extended for other or more general point-to-set operators?*

3.3 Bregman Projections and A Generalized Proximity Function

Next we generalize the proximity function g of (12) or (23) by replacing the orthogonal projections by Bregman projections. Bregman distances and Bregman projections enjoy a large body of published works in the mathematical literature and have been used in diverse fields such as statistics, machine learning, information geometry and optimization, to name but a few.

We denote by $\mathscr{B}(\mathbb{R}^n)$ the set of all Bregman functions $f : \mathbb{R}^n \to \mathbb{R}$ with *zone* $\mathbb{R}^n$. We refer the reader to [17, Chapter 2] for a precise definition of such functions.

For $f \in \mathscr{B}(\mathbb{R}^n)$ the Bregman distance is given by

$$D_f(y,x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle . \tag{48}$$

Consider a nonempty, closed, and convex subset C of $\mathbb{R}^n$. Then the Bregman projection of x onto C with respect to $f \in \mathscr{B}(\mathbb{R}^n)$ is given by

$$P_C^f(x) := \arg\min_{y \in C} D_f(y,x). \tag{49}$$

It is well-known that the Bregman projection exists, and that it is unique when $f \in \mathscr{B}(\mathbb{R}^n)$. We recall the following result for Bregman projections which is [17, Theorem 2.4.2] specialized for $f \in \mathscr{B}(\mathbb{R}^n)$.

Theorem 3.2 *Let $f \in \mathscr{B}(\mathbb{R}^n)$. Then for any $x \in \mathbb{R}^n$, $x^* = P_C^f(x)$ if and only if*

$$\langle \nabla f(x) - \nabla f(x^*), y - x^* \rangle \leq 0, \text{ for all } y \in C. \tag{50}$$

Now

$$D_f(P_C^f(x), x) = f(P_C^f(x)) - f(x) - \left\langle \nabla f(x), P_C^f(x) - x \right\rangle . \tag{51}$$

Lemma 3.1 *Let $f \in \mathscr{B}(\mathbb{R}^n)$, and let C be a nonempty, closed, and convex subset of $\mathbb{R}^n$. Suppose that f is twice continuously differentiable, that $D_f(P_C^f(\cdot), \cdot)$ is a convex function, and that for every hyperplane H the derivatives*

$$\frac{\partial}{\partial x_q}(P_H^f(x))_p, \text{ for } 1 \leq p, q \leq n, \tag{52}$$

exist and are continuous for all $x \in \mathbb{R}^n$, then $D_f(P_C^f(\cdot), \cdot)$ is a continuously differentiable function and

$$\nabla D_f(P_C^f(x), x) = \nabla^2 f(x)(x - P_C^f(x)), \text{ for all } x \in \mathbb{R}^n. \tag{53}$$

Proof. Substitute $S = \mathbb{R}^n$, $\Omega = C$, and $d_C^f(\cdot) = D_f(P_C^f(\cdot), \cdot)$ in [14, Proposition 12]. Then $C \cap \mathrm{cl}(S) \neq \varnothing$ since $C \neq \varnothing$. Moreover, it is clear that f is zone consistent with respect to C, that is, $x \in \mathbb{R}^n$ implies $P_C^f(x) \in \mathbb{R}^n$. Thus, all the conditions of [14, Proposition 12] are satisfied and the result follows. ■

Remark 3.5 *Let $f \in \mathscr{B}(\mathbb{R}^n)$, and let C be a nonempty, closed, and convex subset of $\mathbb{R}^n$. Then $D_f(P_C^f(\cdot), \cdot)$ may not be a convex function. For example, let $C = \{0\} \subset \mathbb{R}$ and $f(x) = (x^2/60) + e^{-x}$. It is clear that f is a strongly convex function with constant $1/30$. Hence, $D_f(y,x) \geq \| y - x \|^2/60$, for any $x, y \in \mathbb{R}$. Therefore, f is a Bregman function with zone $\mathbb{R}$. Since $C = \{0\}$, $P_C^f(x) = 0$ for all $x \in \mathbb{R}$. Thus, $D_f(0,x) = f(0) - f(x) + xf'(x) = 1 + (x^2/60) - (1+x)e^{-x}$. Now, $D_f'(0,x) = (x/30) + xe^{-x}$ and $D_f''(0,x) = (1/30) + (1-x)e^{-x}$. This implies that $D_f''(0,2) = (1/30) - e^{-2} = -0.1020 < 0$. As a result, $D_f(P_C^f(\cdot), \cdot)$ is not convex.*

Unless otherwise stated, by $g_j \in \mathscr{B}(\mathbb{R}^{m_j})$, we mean that $g_j : \mathbb{R}^{m_j} \to \mathbb{R}$ is a Bregman function with zone $\mathbb{R}^{m_j}$, $j = 1, 2, \ldots, t$. We denote $g := (g_j)_{j=1}^t = (g_1, g_2, \ldots, g_t)$.

3.3.1 Our generalized proximity function

Let $f \in \mathscr{B}(\mathbb{R}^n)$ and $g_j \in \mathscr{B}(\mathbb{R}^{m_j})$ for $j = 1, 2, \ldots, t$. By replacing the orthogonal projections by Bregman projections we define the function $p_{f,g} : \mathbb{R}^n \to \mathbb{R}$ by

$$p_{f,g}(x) := \sum_{i=1}^{r} \alpha_i D_f(P^f_{C_i(x)}(x), x) + \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j D_{g_j}\left(P^{g_j}_{Q_h^j(A_j x)}(A_j x), A_j x\right), \ x \in \mathbb{R}^n, \quad (54)$$

where $\alpha_i > 0, \beta_h^j > 0$ are parameters, and C_i, Q_h^j are the point-to-set operators defined above for h, j, i. This is our generalized proximity function.

Remark 3.6 *It is important to note that, in general, a single Bregman function cannot be used to define the relevant Bregman distances in all spaces arising in a split problem. Indeed, the image of $x \in \mathbb{R}^n$ under the bounded linear operator A_j belongs to $\mathbb{R}^{m_j}$ for $j = 1, 2, \ldots, t$. Due to this restriction, we employ a Bregman function $f \in \mathscr{B}(\mathbb{R}^n)$ for nonempty, closed, and convex sets C_i in the domain space and, for each $j = 1, 2, \ldots, t$, a Bregman function $g_j \in \mathscr{B}(\mathbb{R}^{m_j})$ for nonempty, closed, and convex sets Q_h^j for all $h = 1, 2, \ldots, \ell_j$, in the corresponding image space. Thus, the Bregman distances and the Bregman projections are defined in the appropriate underlying Euclidean spaces.*

Proposition 3.7 *Let $f \in \mathscr{B}(\mathbb{R}^n)$ and $g_j \in \mathscr{B}(\mathbb{R}^{m_j})$ for $j = 1, 2, \ldots, t$. The function $p_{f,g} : \mathbb{R}^n \to \mathbb{R}$ of (54) is a generalized proximity function for the problem (MSMS-ICFP).*

Proof. By definition $p_{f,g}(x)$ is non-negative for all $x \in \mathbb{R}^n$. It remains to show that the function $p_{f,g}$ satisfies the property (P_2) of Definition 2.1. Indeed, let $x^* \in \mathbb{R}^n$ be a solution of the problem (MSMS-ICFP). Then using the Bregman projection characterization it is straightforward to verify that

$$x^* = P^f_{C_i(x^*)}(x^*) \text{ and } A_j x^* = P^{g_j}_{Q_h^j(A_j x^*)}(A_j x^*), \text{ for all } h, j, i. \quad (55)$$

This implies that $p_{f,g}(x^*) = 0$.
Conversely, suppose $x^* \in \mathbb{R}^n$ solves the equation $p_{f,g}(x) = 0$. This implies that

$$\sum_{i=1}^{r} \alpha_i D_f(P^f_{C_i(x^*)}(x^*), x^*) + \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j D_{g_j}\left(P^{g_j}_{Q_h^j(A_j x^*)}(A_j x^*), A_j x^*\right) = 0. \quad (56)$$

$$\sum_{i=1}^{r} \alpha_i D_f(P^f_{C_i(x^*)}(x^*), x^*) = 0 \text{ and } \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j D_{g_j}\left(P^{g_j}_{Q_h^j(A_j x^*)}(A_j x^*), A_j x^*\right) = 0. \quad (57)$$

Consequently,

$$\begin{aligned} &\alpha_i D_f(P^f_{C_i(x^*)}(x^*), x^*) = 0, \text{ for all } i = 1, 2, \ldots, r, \text{ and} \\ &\sum_{h=1}^{\ell_j} \beta_h^j D_{g_j}\left(P^{g_j}_{Q_h^j(A_j x^*)}(A_j x^*), A_j x^*\right) = 0, \text{ for all } j = 1, 2, \ldots, t. \end{aligned} \quad (58)$$

Therefore,

$$x^* \in \bigcap_{i=1}^{r} C_i(x^*) \text{ and } A_j x^* \in \bigcap_{h=1}^{\ell_j} Q_h^j(A_j x^*), \text{ for each } j = 1, 2, \ldots, t. \tag{59}$$

Thus, x^* is a solution of the problem (MSMS-ICFP). ■

Remark 3.7 *The analysis presented in the previous sections cannot be directly extended to the general case in the Bregman setting, since it is not well established whether Lemma 2.3 holds for Bregman projections. However, we do make some progress for a special case in the next subsection.*

Various types of Bregman distances and proximity functions were considered in different contexts in the literature. See, e.g., [3, 8, 9].

3.3.2 The split feasibility problem with multiple output sets

An important special case of (MSMS-ICFP) is the split feasibility problem with multiple output sets, which was considered in [39, Subsection 4.1]. We derive here some generalized results about this case by using the generalized proximity function.

Let for all $x \in \mathbb{R}^n$, the point-to-set operator $C_i(x) = C_i$, $i = 1, 2, \ldots, r$, be a nonempty, closed, and convex subset of $\mathbb{R}^n$. Let for all $y \in \mathbb{R}^{m_j}$, the point-to-set operator $Q_h^j(y) = Q_h^j$ be a nonempty, closed, and convex subset of $\mathbb{R}^{m_j}$ for $h = 1, 2, \ldots, \ell_j$, $j = 1, 2, \ldots, t$. Let $A_j : \mathbb{R}^n \to \mathbb{R}^{m_j}$ be a bounded linear operator for each $j = 1, 2, \ldots, t$. Then the problem (MSMS-ICFP) consists of finding a point $x^* \in \mathbb{R}^n$ such that

$$x^* \in \bigcap_{i=1}^{r} C_i \quad \text{and} \quad A_j x^* \in \bigcap_{h=1}^{\ell_j} Q_h^j, \text{ for each } j = 1, 2, \ldots, t. \tag{SFPMOS}$$

The problem (SFPMOS) was studied in [39, Subsection 4.1] for infinite-dimensional Hilbert spaces.

Corollary 3.2 *The function $p : \mathbb{R}^n \to \mathbb{R}$, defined by*

$$p(x) := \frac{1}{2} \sum_{i=1}^{r} \alpha_i \|x - P_{C_i}(x)\|^2 + \frac{1}{2} \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j \|A_j x - P_{Q_h^j}(A_j x)\|^2, \ x \in \mathbb{R}^n, \tag{60}$$

is a proximity function for the problem (SFPMOS).

Proof. The proof is analogous to Proposition 3.2. ■

Let $f \in \mathscr{B}(\mathbb{R}^n)$ and $g_j \in \mathscr{B}(\mathbb{R}^{m_j})$ for $j = 1, 2, \ldots, t$. Define a function $p_{f,g} : \mathbb{R}^n \to \mathbb{R}$ by

$$p_{f,g}(x) := \sum_{i=1}^{r} \alpha_i D_f(P_{C_i}^f(x), x) + \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j D_{g_j}\left(P_{Q_h^j}^{g_j}(A_j x), A_j x\right), \ x \in \mathbb{R}^n. \tag{61}$$

We have the following result.

Corollary 3.3 *Let $f \in \mathscr{B}(\mathbb{R}^n)$ and $g_j \in \mathscr{B}(\mathbb{R}^{m_j})$ for $j = 1, 2, \ldots, t$. The function $p_{f,g} : \mathbb{R}^n \to \mathbb{R}$ defined above in (61) is a generalized proximity function for the problem (SFPMOS).*

Proof. The proof is analogous to Proposition 3.7. ■

Lemma 3.2 *Let $f \in \mathscr{B}(\mathbb{R}^n)$ and C be a nonempty, closed, and convex subset of $\mathbb{R}^n$. Additionally, assume that the function f is σ-strongly convex and ∇f is L-Lipschitz continuous. Then*

$$\|P_C^f(x) - P_C^f(y)\| \leq \frac{L}{\sigma}\|x - y\|, \text{ for all } x, y \in \mathbb{R}^n. \tag{62}$$

Proof. Since $f \in \mathscr{B}(\mathbb{R}^n)$, $P_C^f(x)$ exists and is unique for each $x \in \mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, by Theorem 3.2, we get

$$\left\langle \nabla f(x) - \nabla f(P_C^f(x)), P_C^f(y) - P_C^f(x) \right\rangle \leq 0 \tag{63}$$

and

$$\left\langle \nabla f(y) - \nabla f(P_C^f(y)), P_C^f(x) - P_C^f(y) \right\rangle \leq 0. \tag{64}$$

Adding (63) and (64), we obtain

$$\left\langle \nabla f(P_C^f(y)) - \nabla f(P_C^f(x)), P_C^f(y) - P_C^f(x) \right\rangle \leq \left\langle \nabla f(y) - \nabla f(x), P_C^f(y) - P_C^f(x) \right\rangle. \tag{65}$$

Using the facts that f is continuously differentiable and σ-strongly convex, and ∇f is L-Lipschitz continuous, from (65) we get

$$\sigma\|P_C^f(y) - P_C^f(x)\|^2 \leq L\|y - x\|\|P_C^f(y) - P_C^f(x)\|. \tag{66}$$

If $P_C^f(y) = P_C^f(x)$, the result is immediate. Otherwise, dividing by $\|P_C^f(y) - P_C^f(x)\|$ both sides of (66), we obtain the desired inequality. ■

Corollary 3.4 *Let $f \in \mathscr{B}(\mathbb{R}^n)$ and $g_j \in \mathscr{B}(\mathbb{R}^{m_j})$ for $j = 1, 2, \ldots, t$. Suppose, for each $i = 1, 2, \ldots, r$, the pair (f, C_i) satisfies the hypotheses of Lemma 3.1. Additionally, assume that for each $j = 1, 2, \ldots, t$ and $h = 1, 2, \ldots, \ell_j$, the pair (g_j, Q_h^j) satisfies the hypotheses of Lemma 3.1. Then the function defined in (61) is continuously differentiable and for every $x \in \mathbb{R}^n$, its gradient is given by*

$$\nabla p_{f,g}(x) = \sum_{i=1}^{r} \alpha_i \nabla^2 f(x)(x - P_{C_i}^f(x)) + \sum_{j=1}^{t} \sum_{h=1}^{\ell_j} \beta_h^j A_j^T \nabla^2 g_j(A_j x)(A_j x - P_{Q_h^j}^{g_j}(A_j x)), \tag{67}$$

where $\nabla^2 f$ stands for the Hessian matrix of f and $\nabla^2 g_j$ stands for the Hessian matrix of g_j.

Proof. For each $i = 1, 2, \ldots, r$, the pair (f, C_i) satisfies the hypotheses of Lemma 3.1. Therefore, using Lemma 3.1, we conclude that for each $i = 1, 2, \ldots, r$, $D_f(P^f_{C_i}(\cdot), \cdot)$ is a continuously differentiable function and

$$\nabla D_f(P^f_{C_i}(x), x) = \nabla^2 f(x)(x - P^f_{C_i}(x)), \text{ for all } x \in \mathbb{R}^n. \tag{68}$$

Similarly, for each $j = 1, 2, \ldots, t$ and $h = 1, 2, \ldots, \ell_j$, the pair (g_j, Q^j_h) satisfies the hypotheses of Lemma 3.1. Therefore, using Lemma 3.1, we conclude that for each $j = 1, 2, \ldots, t$ and $h = 1, 2, \ldots, \ell_j$, $D_{g_j}(P^{g_j}_{Q^j_h}(\cdot), \cdot)$ is a continuously differentiable function and

$$\nabla D_{g_j}(P^{g_j}_{Q^j_h}(y), y) = \nabla^2 g_j(y)(y - P^{g_j}_{Q^j_h}(y)), \text{ for all } y \in \mathbb{R}^{m_j}. \tag{69}$$

Now, for each $j = 1, 2, \ldots, t$ and $h = 1, 2, \ldots, \ell_j$, taking $y = A_j x$ in (69) and applying the chain rule, we obtain

$$\nabla D_{g_j}(P^{g_j}_{Q^j_h}(A_j x), A_j x) = A_j^T \nabla^2 g_j(A_j x)(A_j x - P^{g_j}_{Q^j_h}(A_j x)), \text{ for all } x \in \mathbb{R}^n. \tag{70}$$

■

Lemma 3.3 *Assume that all the hypotheses of Corollary 3.4 hold. Assume that for each $i = 1, 2, \ldots, r$, there exists a constant $L^i_1 > 0$ such that*

$$x \to \nabla^2 f(x)(x - P^f_{C_i}(x)) \tag{71}$$

is L^i_1-Lipschitz continuous. Similarly, assume that for each $j = 1, 2, \ldots, t$ and $h = 1, 2, \ldots, \ell_j$, there exists a constant $L^{j,h}_2 > 0$ such that

$$x \to \nabla^2 g_j(A_j x)(A_j x - P^{g_j}_{Q^j_h}(A_j x)) \tag{72}$$

is $L^{j,h}_2$-Lipschitz continuous. Then $\nabla p_{f,g}$ is $\sum_{i=1}^r \alpha_i L^i_1 + \sum_{j=1}^t \sum_{h=1}^{\ell_j} \beta^j_h L^{j,h}_2 \|A_j\|_2$-Lipschitz continuous.

Proof. The proof directly follows from Corollary 3.4 and using the facts that $\sum_{i=1}^r \alpha_i \nabla^2 f(x)(x - P^f_{C_i}(x))$ is $\sum_{i=1}^r \alpha_i L^i_1$-Lipschitz continuous and $\sum_{j=1}^t \sum_{h=1}^{\ell_j} \beta^j_h A_j^T \nabla^2 g_j(A_j x)(A_j x - P^{g_j}_{Q^j_h}(A_j x))$ is Lipschitz continuous with constant

$$\sum_{j=1}^t \sum_{h=1}^{\ell_j} \beta^j_h L^{j,h}_2 \|A_j\|_2.$$

■

Remark 3.8 *It is important to note that we can derive the Lipschitz continuity of the function* $x \to \nabla^2 f(x)(x - P^f_{C_i}(x))$ *using Lemma 3.2 under the assumptions that* $\nabla^2 f$ *is bounded and Lipschitz continuous,* f *is twice continuously differentiable and* σ*-strongly convex, and there exists a constant* $R_i > 0$ *such that*

$$\|x - P^f_{C_i}(x)\| \leq R_i, \text{ for all } x \in \mathbb{R}^n. \tag{73}$$

However, for a fixed nonempty, closed, and convex subset C_i *of* $\mathbb{R}^n$*, the latter assumption is too restrictive. Indeed, it only holds for* $C_i = \mathbb{R}^n$*. To see this, suppose* C_i *is a proper subset of* $\mathbb{R}^n$*. Consider the Bregman function* $f(x) = (1/2)\|x\|^2$*. Then* $P^f_{C_i}(x) = P_{C_i}(x)$*, the orthogonal projection of* x *onto* C_i*. Let* $x \notin C_i$ *and* $p_i = P_{C_i}(x)$*. Assume that* $v_i = x - p_i \neq 0$*. It is observed that* $p_i = P_{C_i}(p_i + tv_i)$*,* $t \geq 0$*, which implies that*

$$dist(p_i + tv_i, C_i) = \|p_i + tv_i - p_i\| = t\|v_i\| \to \infty \text{ as } t \to \infty. \tag{74}$$

This contradicts the assumption in (73).

The following example illustrates the assumptions in (71) and (72) of Lemma 3.3.

Example 3.1 *Let* $f(x) = (1/2)\|x\|^2$*,* $x \in \mathbb{R}^n$*. It is observed that* $\nabla^2 f(x) = I$ *and* $P^f_{C_i}(x) = P_{C_i}(x)$*. Therefore,* $\nabla^2 f(x)(x - P^f_{C_i}(x)) = (I - P_{C_i})(x)$*. It is well-known that* $I - P_{C_i}$ *is nonexpansive for every nonempty, closed, and convex set* C_i*, i.e.,*

$$\|(I - P_{C_i})(x) - (I - P_{C_i})(y)\| \leq \|x - y\|, \text{ for all } x, y \in \mathbb{R}^n. \tag{75}$$

Hence, the assumption in (71) holds with $L^i_1 = 1$*.*

Similarly, assume that $g_j(y) = (1/2)\|y\|^2$*,* $y \in \mathbb{R}^{m_j}$*. Then* $\nabla^2 g_j(y) = I$ *and* $P^{g_j}_{Q^j_h}(y) = P_{Q^j_h}(y)$*. Since* $I - P_{Q^j_h}$ *is nonexpansive for every nonempty, closed, and convex set* Q^j_h*,*

$$\begin{aligned} &\| \ \nabla^2 g_j(A_j x)(A_j x - P^{g_j}_{Q^j_h}(A_j x)) - \nabla^2 g_j(A_j y)(A_j y - P^{g_j}_{Q^j_h}(A_j y))\| \\ &\leq \|A_j(x - y)\| \\ &\leq \|A_j\|_2 \|x - y\|. \end{aligned} \tag{76}$$

Hence, the assumption in (72) holds with $L^{j,h}_2 = \|A_j\|_2$*.*

Remark 3.9 *In addition, let* $C_i = \{x \in \mathbb{R}^n | \|x\| \leq 1\}$ *in Example 3.1, that is, let* C_i *be the closed ball centered at* 0 *with radius* 1*, then*

$$P_{C_i}(x) = \begin{cases} x, & \|x\| \leq 1 \\ \frac{x}{\|x\|}, & \|x\| > 1. \end{cases} \tag{77}$$

Therefore,

$$x - P_{C_i}(x) = \begin{cases} 0, & \|x\| \leq 1 \\ x - \frac{x}{\|x\|}, & \|x\| > 1. \end{cases} \tag{78}$$

Then the assumption in (71) still holds with $L_1^i = 1$. However, for $\|x\| > 1$

$$\|x - P_{C_i}(x)\| = \Big(1 - \frac{1}{\|x\|}\Big)\|x\|, \tag{79}$$

which is unbounded.

Similarly, let

$$A_j = \begin{pmatrix} I_n \\ 0_{(m_j-n)\times n} \end{pmatrix} \in \mathbb{R}^{m_j \times n}, \qquad m_j \geq n,$$

$Q_h^j(y) = \{0\} \subset \mathbb{R}^{m_j}$, *in Example 3.1. Then the assumption in (72) holds with* $L_2^{j,h} = \|A_j\|_2 = 1$. *Moreover,* $P_{Q_h^j}(y) = 0$ *and* $A_j x = (x, 0)$. *Thus,*

$$\|A_j x - P_{Q_h^j}(A_j x)\| = \|A_j x\| = \|x\|, \tag{80}$$

which is unbounded.

Remark 3.10 *We observed that the problem (SFPMOS) is equivalent to the unconstrained minimization problem:*

$$\min_{x \in \mathbb{R}^n} p_{f,g}(x). \tag{81}$$

Algorithm 2 A Generalized Simultaneous Projection Gradient Algorithm (GSPGA) for SFPMOS

1: **Input**: $x^0 \in \mathbb{R}^n$.
2: Compute $L_{\nabla p_{f,g}} := \sum_{i=1}^r \alpha_i L_1^i + \sum_{j=1}^t \sum_{h=1}^{\ell_j} \beta_h^j L_2^{j,h} \|A_j\|_2$.
3: Choose $\lambda \in (0, 2/L_{\nabla p_{f,g}})$.
4: **for** $k = 0, 1, 2, \ldots$ **do**
5: Compute

$$\begin{aligned} x^{k+1} := x^k &- \lambda \sum_{i=1}^r \alpha_i \nabla^2 f(x^k)(x^k - P_{C_i}^f(x^k)) \\ &- \lambda \sum_{j=1}^t \sum_{h=1}^{\ell_j} \beta_h^j A_j^T \nabla^2 g_j(A_j x^k)(A_j x^k - P_{Q_h^j}^{g_j}(A_j x^k)). \end{aligned} \tag{82}$$

Theorem 3.3 *Assume that all the hypotheses of Lemma 3.3 hold. Let the solution set $\widetilde{\Theta}$ of the problem (SFPMOS) is nonempty. Then any sequence $\{x^k\}_{k=0}^\infty$ generated by Algorithm 2 converges in norm to a point in $\widetilde{\Theta}$.*

Proof. Since $\widetilde{\Theta}$ is nonempty, the problem (SFPMOS) is consistent. By Corollary 3.3 and Remark 3.10, the solution set of (SFPMOS) coincides with the global minimizers of the unconstrained minimization problem in (81) and the minimum value of $p_{f,g}$ is zero.

Moreover, by Lemma 3.3, the function $p_{f,g}$ is continuously differentiable, and its gradient is $L_{\nabla p_{f,g}}$-Lipschitz continuous. By the given hypotheses, the function $p_{f,g}$ is convex. Since the step-size $\lambda \in (0, 2/L_{\nabla p_{f,g}})$, all the assumptions of Corollary 2.1 are satisfied. By the expression in (82), our Algorithm 2 is a standard gradient descent algorithm for the problem (SFPMOS). Therefore, by applying Corollary 2.1, we conclude that the sequence $\{x^k\}_{k=0}^{\infty}$ generated by Algorithm 2 converges in norm to a point $x^* \in \arg\min_{x\in\mathbb{R}^n} p_{f,g}(x)$. Since the set of all global minimizers of $p_{f,g}$ coincides with $\widetilde{\Theta}$, $x^* \in \widetilde{\Theta}$. Thus, the sequence $\{x^k\}_{k=0}^{\infty}$ converges in norm to a solution of the problem (SFPMOS). ■

Remark 3.11 *Algorithm 2 is new and distinct from the algorithm studied in [39, Subsection 4.1] for the problem* (SFPMOS).

4 Other Important Special Cases of (MSMS-ICFP)

Specializing from point-to-set operators to single-valued operators brings us back to many split problems, some of which we list here.

1. **The split common fixed point problem**: Let $\mathbb{R}^{m_j} = \mathbb{R}^m$ for all $j = 1, 2, \ldots, t$ and each point-to-set operator $C_i = S_i : \mathbb{R}^n \to \mathbb{R}^n$ be a single-valued nonexpansive operator for each $i = 1, 2, \ldots, r$. Let each point-to-set operator $Q_h^j = \Xi_j : \mathbb{R}^m \to \mathbb{R}^m$ be a single-valued nonexpansive operator for $h = 1, 2, \ldots, \ell_j$, $j = 1, 2, \ldots, t$. Let $A_j = A : \mathbb{R}^n \to \mathbb{R}^m$ be a bounded linear operator for each $j = 1, 2, \ldots, t$. Then the problem (MSMS-ICFP) consists of finding a point $x^* \in \mathbb{R}^n$ such that

$$x^* \in \bigcap_{i=1}^{r} \mathrm{Fix}(S_i) \quad \text{and} \quad A(x^*) \in \bigcap_{j=1}^{t} \mathrm{Fix}(\Xi_j). \tag{SCFPP}$$

   The problem (SCFPP) was studied in [39, Subsection 4.2] for infinite-dimensional Hilbert spaces.

2. **The fixed point problem**: Let $\mathbb{R}^{m_j} = \mathbb{R}^n$ for all $j = 1, 2, \ldots, t$ and each point-to-set operator $C_i = S_i : \mathbb{R}^n \to \mathbb{R}^n$ be a single-valued nonexpansive operator for each $i = 1, 2, \ldots, r$. Assume that $t = r$. Let each point-to-set operator $Q_h^j = C_j$ be a single-valued nonexpansive operator and $A_j = I$, the identity operator, for all $h = 1, 2, \ldots, \ell_j$, $j = 1, 2, \ldots, t$. Then the problem (MSMS-ICFP) consists of finding a point $x^* \in \mathbb{R}^n$ such that

$$x^* \in \bigcap_{i=1}^{r} \mathrm{Fix}(S_i). \tag{FPP}$$

   The problem (FPP) was studied in [39, Subsection 4.3] for infinite-dimensional Hilbert spaces.

3. **The multiple-sets split feasibility problem**: Let $\mathbb{R}^{m_j} = \mathbb{R}^m$, $Q_h^j = Q_j$ be a nonempty, closed, and convex subset of $\mathbb{R}^m$, $A_j = A$ for $h = 1, 2, \ldots, \ell_j$, $j = 1, 2, \ldots, t$, and for $i = 1, 2, \ldots, r$, $C_i = \Omega_i$ be a nonempty, closed, and convex subset of $\mathbb{R}^n$. Then the problem (MSMS-ICFP) reduces to *multiple-sets split feasibility problem* (MSSFP), which was introduced by Censor et al. [11, Subsection 1.1]. That is, the problem (MSMS-ICFP) then consists of finding a point $x^* \in \mathbb{R}^n$ such that

$$x^* \in \Omega := \bigcap_{i=1}^{r} \Omega_i \quad \text{and} \quad Ax^* \in Q := \bigcap_{j=1}^{t} Q_j. \tag{MSSFP}$$

4. **The split feasibility problem**: Let $\mathbb{R}^{m_j} = \mathbb{R}^m$, $Q_h^j = Q$ be a nonempty, closed, and convex subset of $\mathbb{R}^m$, $A_j = A$ for $h = 1, 2, \ldots, \ell_j$, $j = 1, 2, \ldots, t$, and for $i = 1, 2, \ldots, r$, $C_i = C$ be a nonempty, closed, and convex subset of $\mathbb{R}^n$. Then the problem (MSMS-ICFP) reduces to the classical *split feasibility problem* (SFP), which was first introduced and studied by Censor et al. [12, Section 1 and Subsection 6.1]. That is, the problem (MSMS-ICFP) then consists of finding a point $x^* \in \mathbb{R}^n$ such that

$$x^* \in C \quad \text{and} \quad Ax^* \in Q. \tag{SFP}$$

5. **The convex feasibility problem**: Let $t = r$, $\mathbb{R}^{m_j} = \mathbb{R}^n$, $Q_h^j = \Omega_j$ be nonempty, closed, and convex subsets of $\mathbb{R}^n$, $A_j = I$, the identity operator on $\mathbb{R}^n$, for $h = 1, 2, \ldots, \ell_j$, $j = 1, 2, \ldots, t$, and for $i = 1, 2, \ldots, r$, $C_i = \Omega_i$ be nonempty, closed, and convex subsets of $\mathbb{R}^n$. Then the problem (MSMS-ICFP) reduces to *convex feasibility problem* (CFP). That is, the problem (MSMS-ICFP) then consists of finding a point $x^* \in \mathbb{R}^n$ such that

$$x^* \in \Omega := \bigcap_{i=1}^{r} \Omega_i. \tag{CFP}$$

   The problem (CFP) was studied by many researchers (see, for example, [2] and [17, Chapter 5]).

6. **The split common minimization problem:** Let $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i = 1, 2, \ldots, r$, and $g_j : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$, $j = 1, 2, \ldots, t$ be proper, lower semi-continuous, and convex functions. Let $\mathbb{R}^{m_j} = \mathbb{R}^m$ and $A_j = A : \mathbb{R}^n \to \mathbb{R}^m$ be a bounded linear operator for all $j = 1, 2, \ldots, t$. If we set $C_i(x) = \arg\min_{u \in \mathbb{R}^n} f_i(u) \neq \varnothing$, $i = 1, 2, \ldots, r$, $Q_h^j(y) = \arg\min_{v \in \mathbb{R}^m} g_j(v) \neq \varnothing$, $h = 1, 2, \ldots, \ell_j$, $j = 1, 2, \ldots, t$, then the problem (MSMS-ICFP) reduces to the following minimization problem: find $x^* \in \mathbb{R}^n$ such that

$$x^* \in \bigcap_{i=1}^{r} \arg\min_{u \in \mathbb{R}^n} f_i(u) \quad \text{and} \quad Ax^* \in \bigcap_{j=1}^{t} \arg\min_{v \in \mathbb{R}^m} g_j(v). \tag{83}$$

   This problem is called the *split minimum point problem* in [40, Subsection 4.1], where it is studied in infinite-dimensional Hilbert spaces.

5 Conclusion

We introduced the multi-split multiple-sets implicit convex feasibility problem (MSMS-ICFP), which provides a general framework for feasibility-seeking problems involving implicit constraint sets in the domain space and multiple split implicit constraint sets in possibly different image spaces. The formulation extends the implicit convex feasibility problem and many other well-known inverse split problems. We employed a proximity function to reformulate the problem (MSMS-ICFP) as an unconstrained minimization problem. We analyzed various results on the properties of the proximity function and established convergence of the sequence generated by the proposed simultaneous projection gradient method in the consistent case. Additionally, we presented a Bregman-distance-based generalization. Since the domain and image constraints belong to different Euclidean spaces, the generalization uses appropriate Bregman functions in the relevant spaces. For the split feasibility problem with multiple output sets, this led to a generalized simultaneous projection gradient method. We established norm convergence of the sequence generated by the algorithm in the consistent case. The special cases of the problem (MSMS-ICFP) are presented in this work to illustrate the scope of the proposed framework and its relation to various existing models in feasibility, fixed point theory, and split inverse problems. Moreover, they show that the problem (MSMS-ICFP) can serve as a common setting for studying apparently different problems. During our analysis, a number of open research questions emerge. And, of course, future numerical experiments and applications would be useful for demonstrating the practical performance of the proposed methods.

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